Karlsruher Institut für Technologie

Masters Thesis

Non-Negative Sectional Curvature Metrics on Exotic Seven-Spheres

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Expressing gratitude for the profound privilege of consciousness, through which the universe comes to know itself.

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Chapter 1

Introduction

The interplay between topology and geometry has long been a central focus of mathematical study. A particularly fascinating area lies in understanding the constraints that topological structures place on Riemannian curvature properties. Mainly, there are three ways of describing and measuring curvature on Riemannian manifolds: sectional, Ricci, and scalar curvature. Among those, sectional curvature is the strongest, Ricci curvature being its average, and scalar curvature the trace of the Ricci curvature tensor. Sectional curvature of a tangent surface σ at a point p is motivated by the well-known Gaussian curvature of a surface generated by geodesics starting at p and with tangents in σ.

Definition 1.1 (3.1.3 [36])**.** Let (M, g) be a Riemannian manifold. Given a point $p \in M$ and a two-dimensional subspace σ of the tangent space T_pM at p, the real number

$$sec(\sigma) := K(x, y) := \frac{g(R(y, x)x, y)}{g(x, x)g(y, y) - g(x, y)^2},$$

where $x, y \in \sigma$ are linearly independent, R is the curvature tensor (cf. 3.1.1 [36]), is called sectional curvature of σ at p and does not depend on the choice of x and y.

In particular, one is interested in classifying manifolds with respect to a fixed sign of their curvatures, i.e., non-negatively, non-positively, or in the strict cases of positively or negatively curved manifolds. Non-positively curved, complete, connected manifolds are determined (up to diffeomorphism) by their fundamental group π_1 due to the famous Hadamard-Cartan Theorem.

Theorem 1.2 (Hadamard-Cartan 6.2.2 [36])**.** *Let (M, g) be a connected, complete Riemannian manifold of non-positive sectional curvature. Then the universal covering of M is diffeomorphic to $\mathbb{R}^n$, where $n = dim(M)$. Thus, M is diffeomorphic to $\mathbb{R}^n/\pi_1(M)$ and especially if M is simply connected, it is diffeomorphic to $\mathbb{R}^n$.*

For non-negatively curved manifolds, the classification reduces to the compact case by the Soul Theorem of Cheeger and Gromoll.

Theorem 1.3 ([6])**.** *Suppose (M, g) is a complete, connected, non-compact Riemannian manifold with non-negative sectional curvature. In that case, there exists a compact totally geodesic submanifold $S \subseteq M$, the soul of M, such that M is diffeomorphic to the normal bundle of S in M.*

Another interesting result directly relating topology and curvature is the quarter-pinched Sphere Theorem.

Theorem 1.4 ([4], [23]). *Let (M, g) be a complete, simply connected Riemannian manifold with*

$$0 < \frac{1}{4}k < sec(\sigma) \leq k, \ \forall \sigma \ tangent \ plane$$

and a fixed number k, then M is homeomorphic to $S^n, n = dim(M)$.

In contrast to non-existence results given by these theorems, it is harder to prove the existence of a non-positive or non-negatively curved metric. In this thesis, we aim to examine a simply connected, compact case, namely, manifolds homeomorphic to the standard sphere. If a manifold is homeomorphic but not diffeomorphic to the standard sphere, we call it an exotic sphere. Thus, by the Hadamard-Cartan Theorem, our interest in finding metrics on exotic spheres is directed to non-negative or even strictly positive curvature. It is of interest to investigate metrics on exotic spheres to determine whether curvature is sensitive to smooth structures. With the standard round metric, the standard sphere is strictly positively curved. There is a further restriction on the curvature of exotic spheres, given by a stronger version of the quarter-pinched Sphere Theorem above.

Theorem 1.5 ([39]). *Let (M, g) be a complete, simply connected Riemannian manifold with*

$$\delta \leq sec(\sigma) \leq 1, \ \forall \sigma \ tangent \ plane$$

and $\delta = 0.681$, then M is diffeomorphic to $S^n, n = dim(M)$.

Therefore, in particular, exotic spheres can not admit such a pinched metric. Remarkably, all known examples of simply connected, compact manifolds with strictly positive curvature appear as biquotients of Lie groups, i.e., as manifolds diffeomorphic to $(G/H)/K$, where G is a compact Lie group with K and H closed subgroups, such that K acts freely on G/H (cf. [41]). Further note that every biquotient has a metric of non-negative sectional curvature induced by the one on compact Lie groups (Chapter 2, Proposition 6.6 [18], Proposition 4.4.1 & 4.4.2 [36], and O'Neil's formula 2.14). Gromoll and Meyer constructed the first metric with non-negative curvature on an exotic sphere in dimension seven ([11]), which is a biquotient. Around 30 years later, Wilhelm proved the existence of a positively curved metric almost everywhere ([43]). Still, no such result holds in general for exotic spheres, and the Gromoll-Meyer sphere is the only one in dimension seven appearing as a biquotient. At least, metrics of non-negative sectional curvature on every exotic seven-sphere exist, which is the topic of this thesis.

It was rather unexpected when John W. Milnor first proved the existence of exotic spheres in 1956 ([30]). He discovered those by considering boundaries of certain eight-dimensional manifolds, which appear as rank-four disk bundles over S^4. S^4 can be viewed as gluing two hemispheres D^4 along their boundary S^3 via a map

$$f_{i,j} : S^3 \to SO(4), f_{i,j}(u)(v) = u^i v u^j,$$

where S^3 is understood as the unit quaternions. This map is called the clutching function, and one may classify the disk bundles by tuples $(i, j) \in \mathbb{Z} \oplus \mathbb{Z} = \pi_3(SO(4))$ (cf. C.2). Further, Milnor defined an invariant λ for oriented, closed, seven-dimensional manifolds M^7, with $H^3(M^7) = H^4(M^7) = 0$ and eight-dimensional, compact, oriented coboundaries W^8 (it can be shown that indeed such coboundaries exist, cf. [40]). With these assumptions, one gets by the long exact sequence (B.5) that $j : H^4(W^8, M^7; \mathbb{Q}) \xrightarrow{\cong} H^4(W^8; \mathbb{Q})$ is an isomorphism. The definition of λ is then based on the Hirzenbruch Signature Theorem (cf. second equation in 3.43) and given by

$$\lambda(M^7) := 2j^{-1}p_1(W^8) - sig(W^8, M^7) \bmod 7,$$

where $p_1(W^8)$ is the first rational Pontryagin class of W^8 and $sig(W^8, M^7)$ denotes the signature of the pullback under j of the rational quadratic form, i.e., of

$$\alpha \to \alpha \smile \alpha, \alpha \in H^4(W^8; \mathbb{Q})$$

evaluated on the pullback of the fundamental class $[W^8]$ of W^8. λ can be shown to be independent of the choice of the coboundary W^8. This invariant can be used to prove indeed the existence of 20 exotic structures appearing as S^3-bundles over S^4 – nowadays known as the Milnor spheres. Three years later, Milnor came up with the definition of a group θ_n of equivalence classes of homotopy spheres in dimension n ([29]). At around the same time, Smale proved the generalized Poincaré conjecture in dimension greater than or equal to five.

Theorem 1.6 (Generalized Poincaré's Conjecture [37])**.** *A closed n-dimensional manifold homotopy equivalent to S^n is homeomorphic to S^n for $n > 4$.*

Therefore, manifolds homotopic to S^n and of dimension greater than four are homeomorphic to the standard sphere (dimension two was proven in the early 20s, dimension four by Freedman in 1982, and dimension three was proven in 2020 by Perelman). The groups θ_n consist now of $h-$cobordant equivalence classes of homotopy $n-$spheres, i.e., of manifolds homeomorphic to S^n (in the original paper, it is denoted as J-equivalence).

Definition 1.7. Two differentiable oriented manifolds M, N are called h-cobordant if there exists a differentiable oriented manifold W such that $M \sqcup -N = \partial W$ and M, N are deformation retracts of W.

Smale further proved that, in dimensions greater than four, h-cobordism means diffeomorphism ([38]). To obtain a group out of the h-cobordism equivalence classes of homotopy spheres, Milnor proved the following.

Theorem 1.8 ([30])**.** *The connected sum operation makes θ_n into a well-defined abelian group.*

Milnor and Kervair completely determined the groups θ_n for small $n \neq 3$ (up to $n = 18$) (cf. [22]) and, with their proofs, laid the foundation of modern surgery theory. It turns out that all groups are finite, and dimension seven is the first one

where the group is not trivial (for the case $n = 3$ cf. [33]). θ_7 is cyclic of order 28, and the generator is the earlier mentioned Gromoll-Meyer sphere. In 1976, Gromoll and Meyer constructed this sphere as a biquotient of $Sp(2)$, which gave them the possibility to prove the existence of a non-negative sectional curvature metric on it ([11]). Grove and Ziller first found such metrics for every Milnor sphere in 2000 ([13]) based on properties of cohomogeneity one manifolds (cf. Chapter 4 and Section 2.1). This construction was the basis for Goette, Kerin, and Shankar in 2020 to construct metrics of non-negative sectional curvature on every exotic seven-sphere ([10], cf. Section 5.1). In fact they proved the more general statement that there is a six-parameter family of seven-dimensional manifolds $M^7_{\underline{a},\underline{b}}$ which are two-connected and have forth (integer) cohomology $\mathbb{Z}_n, n$ depends on $\underline{a}, \underline{b}$, such that for certain $\underline{a}, \underline{b}$ they have an $SO(3)-$invariant metric of non-negative sectional curvature. Though $SO(3)$-invariance is not important in this thesis, it is worth considering when discussing symmetry. In this thesis, we concentrate on the case where the $M^7_{\underline{a},\underline{b}}$ are homotopy spheres. For the Milnor spheres, one could use their appearance as S^3-bundles over S^4 to prove that the manifolds constructed by Grove and Ziller are diffeomorphic to them; this is not possible for the non-Milnor spheres. Therefore Goette, Kerin, and Shankar used the so-called Eells-Kuiper invariant μ, defined by Eells and Kuiper in 1962 ([8]). In contrast to the invariant λ, μ is useful for a broader class of manifolds M. Namely manifolds of dimension $4k-1$ that are closed, smooth, oriented, two-connected, and admit a compact, smooth, oriented spin coboundary W (still such a spin coboundary always exist in dimension seven, cf. 3.25), such that $j : H^{2k}(W, M; \mathbb{Q}) \xrightarrow{\cong} H^{2k}(W; \mathbb{Q})$ and $j : H^{4i}(W, M; \mathbb{Q}) \xrightarrow{\cong} H^{4i}(W; \mathbb{Q}), 0 < i < k$ are isomorphisms (especially this assumptions hold for the family $M^7_{\underline{a},\underline{b}}$ mentioned above). The Eells-Kuiper invariant is constructed analogously to λ, but now based on the Index Theorem 3.42, and defined in dimension seven as (see Section 5.3.1)

$$\mu(M^7) = \frac{1}{2^7 \cdot 7}((j^{-1}p_1(W^8))^2 - 4sig(W^8, M^7)) \bmod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}.$$

Whereas the definition is straightforward, its computation is complex and requires some foreknowledge. This work aims to rigorously structure the background knowledge from Spin Geometry, enabling a deeper understanding of the classification of diffeomorphism types of exotic seven-spheres and the induced metrics on them.

This thesis is structured into five chapters and three appendices, which provide the necessary background on fiber bundles, algebraic topology, and characteristic classes, respectively. Chapter 2 should be understood as a reminder or a brief introduction to topics of Riemannian geometry. First (Section 2.1), we introduce Cohomogeneity one manifolds, which play a crucial role in Chapter 4. Riemannian submersions and foliations (Section 2.2) are fundamental in the whole thesis, and especially O'Neill's curvature formula for submersions 2.14 is of great importance. We lastly introduce orbifolds and Seifert fibrations in Section 2.3, since it will turn out that exotic seven-spheres appear as total spaces of a Seifert fibration over an orbifold (Section 5.3.2). Chapter 2 primarily highlights how the individual definitions relate to one another. In Chapter 3 we discuss several topics in Spin Geometry. We start with Clifford Algebras and the groups Pin and Spin (Section 3.1), which are fundamental to defining spin manifolds (Section 3.2) and spinor bundles (Section 3.3).

Afterward, we introduce the Dirac operator (Section 3.4) (on spinor bundles). This first-order differential operator plays an essential role in defining and calculating the Eells-Kuiper invariant in Chapter 5.3, since they are related via Index Theorems, handled in Section 3.4.3. Chapter 4 introduces the construction of non-negatively curved metrics on every Milnor sphere by Grove and Ziller. Based on that, we describe in Section 5.1 how this result can be generalized to a family of manifolds $M^7_{\underline{a},\underline{b}}$. Section 5.2 proves that for certain $\underline{a}, \underline{b}$, these manifolds are indeed homeomorphic to S^7. In the last Section 5.3 all the results from Chapter 3 are finally applied to define the Eells-Kuiper invariant and to prove that it distinguishes different exotic-seven spheres (Section 5.3.1). Lastly, in Section 5.3.2, we explain how this invariant can be computed, to conclude that every exotic seven-sphere is attained as a manifold $M^7_{\underline{a},\underline{b}}$ and thus admits a metric of non-negative sectional curvature.

The reader is expected to be familiar with the fundamentals of differential geometry, topology, and Lie theory. If not mentioned otherwise, all manifolds are assumed to be smooth, complete, connected, and without boundary.

Chapter 2

Preliminaries from Riemannian Geometry

This chapter introduces topics in Riemannian geometry that go beyond basic knowledge and are essential throughout this thesis. Section 2.1 discusses cohomogeneity one manifolds, which are generalizations of homogeneous spaces. A Riemannian manifold (M, g) is homogeneous if its isometry group acts transitively on M, i.e., "looks the same" at each point. Suppose M admits a smooth transitive action by a compact Lie group. In that case, there always exists a metric on M such that the action becomes isometric (this holds in general for an action by a compact Lie group, cf. Theorem 1.13 [32]). The definition of a cohomogeneity one manifold is a natural extension of the concept, in which the action of a compact Lie group is not transitive, but the orbit space is one-dimensional. We show that cohomogeneity one manifolds admit a decomposition into bundles over orbits given by the Slice Theorem 2.8. It will be helpful in Chapter 4 for constructing the Milnor spheres as well as in the preceding sections for discussing the structure of certain cohomogeneity one manifolds. Cohomogeneity one manifolds can also be described as Riemannian foliations, handled in Section 2.2. We will be able to describe exotic seven-spheres as foliations in Chapter 5. Riemannian foliations generalize Riemannian submersions. Whereas smooth submersions do not require a metric, Riemannian submersions depend heavily on a metric. For example, all fiber bundles are smooth submersions. Riemannian submersions often appear as quotient maps associated with a compact Lie group action. One can think of it as isometrically preserving all the directions orthogonal to the orbits and forgetting the others. An essential result on Riemannian submersions is O'Neill's Formula 2.14, which relates the curvatures of the domain and target of the map. It plays a crucial role when proving non-negative curvature in the Chapters 4 and 5. In the last Section 2.3 of this chapter, we introduce Orbifolds and Seifert fibrations. Orbifolds are a generalization of manifolds. They allow the space to have singularities, for example, the tip of a cone. A finite group — the isotropy group — describes its symmetries in the neighborhood of each point. A Seifert fibration is a bundle over an orbifold, called an orbi-bundle, such that it is a Riemannian foliation. As already mentioned, exotic seven-spheres can be described as Riemannian foliations, so we can also view them as Seifert fibrations. This point of view has application when calculating the Eells-Kuiper invariant in Section 5.3.2.

2.1 Cohomogeneity One Manifolds

If not specified, this section is based on [13].

Definition 2.1. Let M be a closed manifold. It is of cohomogeneity one if there exists an action of a compact Lie group G on M such that the orbit space M/G is one-dimensional.

Recall that M/G is indeed a manifold (Theorem 21.10 in [28]). Further, since compact Lie groups admit the Haar measure, one can average over a given metric on M to obtain a G invariant metric (cf. Theorem 1.13 [32]).

Remark 2.2. If a closed manifold M has a finite fundamental group $\pi_1(M)$ and a cohomogeneity one action by a compact Lie group G, then M/G is a closed interval.

Proof. Since M/G is one-dimensional, it follows from the closeness of M that M/G is either a circle or an interval. Because $\pi_1(M)$ is finite, the homotopy sequence of $M \to M/G$ (B.12) implies that $\pi_1(M/G)$ is finite and thus M/G cannot be a circle. □

Since later in this thesis we are only interested in cohomogeneity one manifolds with a finite fundamental group, namely, homotopy spheres, we restrict ourselves to that case. One standard example is the following.

Example 2.3. Consider the natural $SO(n)$ action on the n-sphere S^n. $SO(n)$ acts via matrix multiplication. For this, embed $SO(n)$ into $SO(n+1)$ by adding a one on the diagonal. The position of the one describes the axis around which S^n is rotated. For illustrative purposes, we consider the case $n = 2$.

$$\rho : SO(2) \times S^2 \to S^2, (A, v) \to \begin{pmatrix} A & \\ & 1 \end{pmatrix} v$$

defines the described action. Then $S^2/SO(2) \cong [0, \pi]$ and thus S^2 (and in general other n-spheres) is of cohomogeneity one. Figure 2.1 illustrates the action where the red lines demonstrate the orbits and should be thought to be continuously extended throughout the sphere, and $\pi : S^2 \to S^2/SO(2)$ is the quotient map. Note that $SO(n+1)$ acting on S^n by multiplication (illustrated by the blue arrows) is a transitive action and thus S^n with the standard metric is also homogeneous.

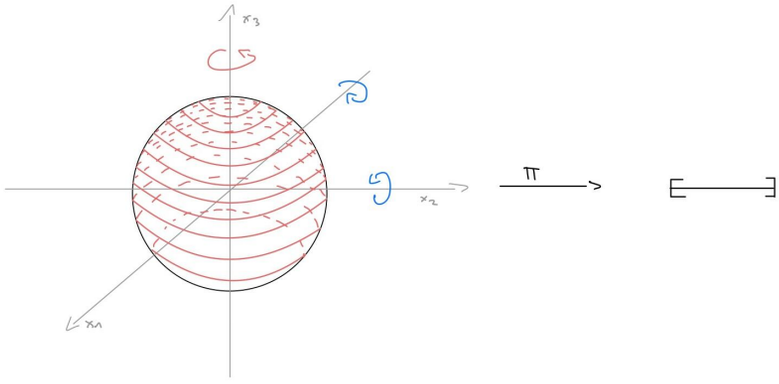


Figure 2.1: $SO(2)$ acting on S^2

Looking at Figure 2.1, one can immediately see that some orbits under the cohomogeneity one action are larger than others. In particular, the orbits corresponding to the south and north poles are a single point, whereas the other orbits are great circles. To make this precise, we introduce the following definitions.

Definition 2.4. Let G be a Lie group acting on a manifold M. The Lie subgroup $G_x := \{g \in G | gx = x\}, x \in M$ is called the isotropy group of x. Note, that $G/G_x \cong Gx$ where Gx is the orbit of x under G.

Remark 2.5. If two elements lie in the same orbit, then their isotropy groups are conjugate.

Proof. Let x, y be in the same orbit, then there exists an element $g \in G$ such that $gx = y$. It follows that

$$y = gx = gG_x x = gG_x g^{-1} y \text{ and } x = g^{-1}y = g^{-1}G_y y = g^{-1}G_y gx,$$

implying $G_y = gG_x g^{-1}$. □

Definition 2.6. Let K, H be two isotropy groups, then define

$$H \succeq K :\Leftrightarrow \exists g \in G : H \subseteq gKg^{-1}.$$

Motivated by Remark 2.5, we say that K and H are of the same orbit type if $K \preceq H$ and $H \preceq K$.

Theorem 2.7 (3.1 [5])**.** *If G is a compact Lie group acting on a manifold M, then there exists a unique maximal orbit type H with respect to $\succeq$, called the principal type for M. The union of the orbits of this type is a dense and open subset in M, and the image in M/G is connected.*

Note that maximal orbit type refers to a maximal size of the orbit and not the isotropy group. For the rest of this section, let M be as in Remark 2.2. We can now conclude with Theorem 2.7 that M has at most three orbit types, namely at most two singular types corresponding to the endpoints of the interval and the principal type (the quotient map is open!). In Example 2.3, the isotropy groups of the north and south poles are both $SO(2)$ and of singular type. All other isotropy groups consist only of the identity and are of the principal type.

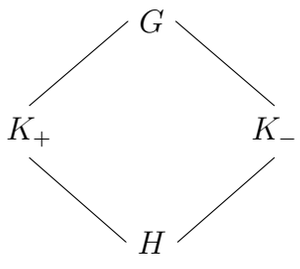


Figure 2.2: Diagram of a cohomogeneity one manifold

With this observation, we can now conclude that the cohomogeneity one action on M determines the following Diagram 2.2. Let G be the Lie group with cohomogeneity one action on M. $K_\pm$ denote the (conjugacy class of the) isotropy groups of singular types, and H the isotropy group of principal type. Recall that $H \subset K_\pm \subseteq G$. One further has that $K_\pm/H = S^{l_\pm}, l_\pm \leq dim(G)$ (Theorem 2 in [34]).

Conversely, given groups $K_\pm, H$ with $H \subset K_\pm \subseteq G$ and $K_\pm/H = S^{l_\pm}$, such a diagram also determines a cohomogeneity one manifold. This is explained in the following. For the definitions of fiber and associated bundles, see Appendix A.

Theorem 2.8 (Slice Theorem 5.6.24 [36]). *Let (M, g) be a Riemannian manifold and H a closed subgroup of the isometry group of (M, g). Then for every $p \in M$ there exists a map $\exp^{\perp} : H \times_{H_p} T_p^{\perp} Hp \to M$ that is a diffeomorphism from $H \times_{H_p} B(0, \epsilon)$ on to an $\epsilon-$ neighborhood of the orbit Hp.*

Again, let M, G be as in Remark 2.2 and $\pi : M \to M/G \cong [-1, 1]$ be the quotient map. We fix a point $x_0 \in \pi^{-1}(0)$. Let $c : [-1, 1] \to M$ be the unique geodesic with $c(0) = x_0$ and $c \circ \pi = id$. c therefore intersects each orbit orthogonally, and its tangents are normal to the orbits. Define $x_{\pm} := \pi^{-1}(\pm 1)$, and denote the isotropy groups of $x_{\pm}$ by $K_{\pm}$ and of x_0 by H. Then $D_- := \pi^{-1}([-1, 0])$ and $D_+ := \pi^{-1}([0, 1])$ are tubular neighborhoods of the singular orbits $Gx_{\pm}$ and with the Slice Theorem we get $D_{\pm} \cong G \times_{K_{\pm}} D^{l_{\pm}}$, where $D^{l_{\pm}+1} \subset N(Gx_{\pm})$ is the normal unit disk (if the orbit space is not normed to $[-1, 1]$ we need to reparametrize first, or describe the following with respect to ϵ neighborhoods, which is topologically the same). This means that we can describe M as the union of two disk bundles glued along their boundary, the principal orbit.

$$M = G \times_{K_-} D^{l_-+1} \cup_{G/H} G \times_{K_+} D^{l_++1}. \tag{2.1}$$

Notice that $G/H = \partial D_{\pm} = G \times_{K_{\pm}} S^{l_{\pm}}$. So, given a compact Lie group G and closed subgroups $H \subset K_{\pm} \subseteq G$ with $K_{\pm}/H = S^{l_{\pm}}$, one can recover a unique (up to G-equivariant diffeomorphism, i.e., up to a diffeomorphism $f : M \to M'$ with $f(gx) = gf(x)$) cohomogeneity one manifold M by construction (2.1) and given by Diagram 2.2.

Example 2.9 (Continue Example 2.3). Let x_0 lie on the equator of S^2 and c be the geodesic through x_0 illustrated by the dark blue line in Figure 2.3. The south and north poles give $x_{\pm}$. The normal disks corresponding to $K_{\pm} \cong SO(2)$ are

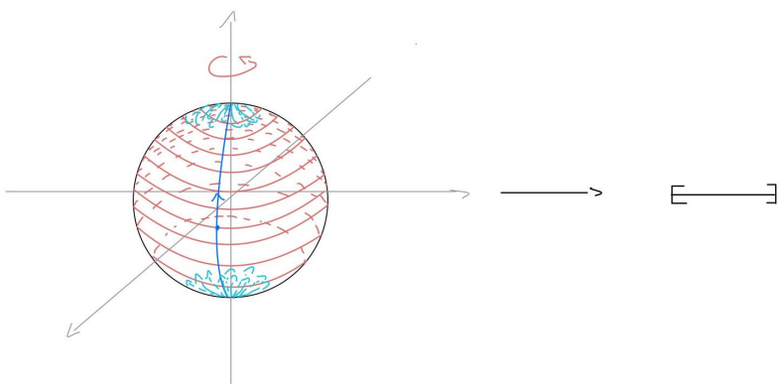


Figure 2.3: Slice decomposition of S^2 by $SO(2)$.

illustrated by the light blue arrows. The upper and lower hemispheres are tubular neighborhoods of the north and south poles, respectively, and indeed given by $SO(2) \times_{SO(2)} D^2 \cong D^2$. Further $SO(2) \cong S^1$ so the decomposition (2.1) reads for this example as

$$S^2 = D^2 \cup_{S^1} D^2.$$

Grove and Ziller further invented a construction of principal bundles over cohomogeneity one manifolds. This construction will be applied in Section 4 to obtain a non-negative sectional curvature metric on principal S^3-bundles over S^4, so in particular, on Milnor spheres.

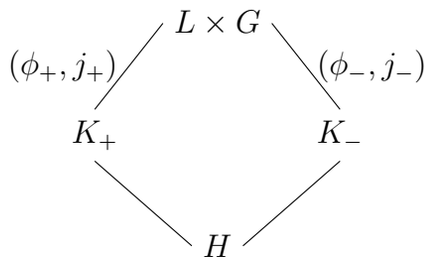


Figure 2.4: Diagram of a principal $L-$bundle

Theorem 2.10. *Let L be a compact Lie group and M as before with group diagram 2.2. Let $j_\pm : K_\pm \to G, \iota_\pm : H \to K_\pm$ denote the inclusions. If there exist Lie group homomorphisms $\phi_\pm : K_\pm \to L$, $\phi_0 : H \to L$ with $\phi_\pm \circ \iota_\pm = \phi_0$, then the cohomogeneity one manifold P defined by the diagram 2.4 is a principal $L-$bundle over M.*

Proof. If we view $K_\pm, H$ via the homomorphisms

$$(\phi_\pm, j_\pm) : K_\pm \to L \times G, k \quad \to \quad (\phi_\pm(k), j_\pm(k))$$

as subgroups of $L \times G$, it holds that

$$\begin{aligned} L \cap (l,g)K_\pm(l,g)^{-1} &= (l,g)(L \times \{e\} \cap K_\pm)(l,g)^{-1} \\ &= \{(ll'l^{-1}, e) | l' \in L \cap \phi_\pm(K_\pm), (l', e) \in \phi_\pm(K_\pm) \times j_\pm(K_\pm)\} \\ &= \{(lel^{-1}, e)\} \\ &= \{(e,e)\} \\ L \cap (l,g)H(l,g)^{-1} &= (l,g)(L \times \{e\} \cap H)(l,g)^{-1} \\ &= \{(e,e)\}, \quad \forall (l,g) \in L \times G. \end{aligned}$$

So $L = L \times \{e\}$ is not conjugated to any subgroup of the isotropy groups and thus the subaction of $L \times G$ by $L \times \{e\}$ on P is free. Furthermore, this implies that $P/L = M$ since they have the same cohomogeneity one diagram. Thus, P is a principal $L-$bundle over M. □

2.2 Riemannian Submersions and Foliations

This section is based on Chapter 9 in [3] and [12].

Definition 2.11. A smooth map $\pi : \tilde{M} \to M$ between two manifolds is called a submersion if its differential is surjective at each point. Assuming so, and that $\tilde{M}, M$ are equipped with Riemannian metrics $\tilde{g}, g$, let $q \in \pi^{-1}(p)$, and $p \in M$. Define

$$\mathcal{V}_q := T_q\pi^{-1}(\pi(q)) = T_q\pi^{-1}(p) = ker(d\pi_q) \text{ and } \mathcal{H}_q := \mathcal{V}_q^\perp$$

with respect to $\tilde{g}$ then $T_q\tilde{M}$ splits into the orthogonal sum $T_q\tilde{M} = \mathcal{V}_q \bigoplus \mathcal{H}_q$. π is called a Riemannian submersion if $d\pi_q : \mathcal{H}_q \to T_{\pi(q)}M$ is isometric $\forall q \in \tilde{M}$. We call $\mathcal{V}_q$ vertical and $\mathcal{H}_q$ horizontal subspace at q. The vertical $\mathcal{V}_q(v)$ or horizontal component $\mathcal{H}_q(v)$ of a tangent vector $v \in T_q\tilde{M}$ (or of a vector field) is the restriction $v|_{\mathcal{V}_q}$, $v|_{\mathcal{H}_q}$ of it onto the respective vertical or horizontal subspaces.

Theorem 2.12 (9.12 [3]). *Let (M, g) be a Riemannian manifold and G a closed subgroup of the isometry group of (M, g). Suppose $\pi : M \to M/G$ is a smooth submersion (this is especially the case if G is a compact Lie group acting freely). In that case, there exists one and only one Riemannian metric $\tilde{g}$ and a manifold structure on M/G such that π is a Riemannian submersion.*

Note that if π is a smooth submersion that is surjective, it is a quotient map (cf. Proposition 4.28 [28]).

Definition 2.13. Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. Let X be a vector field on M. We call a vector field $\tilde{X}$ on $\tilde{M}$ with vertical component $\mathcal{V}(\tilde{X}) = 0$ and $d\pi(\tilde{X}) \equiv X \circ \pi$ the unique horizontal lift of X.

Theorem 2.14 (O'Neills formula [35]). *If $\pi : \tilde{M} \to M$ is a Riemannian submersion and X, Y orthonormal vector fields on M, it holds that*

$$K(X, Y) \circ \pi \equiv \tilde{K}(\tilde{X}, \tilde{Y}) + \frac{3}{4} \parallel \mathcal{V}([\tilde{X}, \tilde{Y}]) \parallel^2$$

Example 2.15 (Based on 1.1.5 [27]). Consider the Hopf Fibration

$$\begin{aligned} S^1 \to S^3 \xrightarrow{h} S^2 \cong \mathbb{C}P^1 \\ (z, w) \to [\frac{z}{w} : 1], \end{aligned}$$

where we view S^3 as a subset of $\mathbb{C}^2$ equipped with the standard Hermitian inner product and S^2 with the Fubini-Study metric. Note, that the fiber containing (z, w) also contains elements of the form $(e^{i\theta}z, e^{i\theta}w), \theta \in [0, 2\pi)$. This implies that the vector tangent to the fiber at (z, w) is given by $\frac{\partial}{\partial\theta}|_{\theta=0}(e^{i\theta}z, e^{i\theta}w) = i(z, w)$. The vectors $\lambda(-\bar{w}, \bar{z}), \lambda \in \mathbb{C}$ form the (real two-dimensional) horizontal subspace at (z, w) :

$$\begin{aligned} \langle i(z, w), \lambda(-\bar{w}, \bar{z})\rangle &= \sqrt{((iz - \bar{\lambda}\bar{w})^2 + (iw\bar{\lambda}\bar{z})^2} \\ &= \bar{\lambda}\sqrt{z^2w^2 - w^2z^2} \\ &= 0 \end{aligned}$$

To show that h is a Riemannian submersion we need to show that

$$Dh|_{(z,w)}(\lambda(-\bar{w}, \bar{z})) \ \in \ T_{h((z,w))}S^2$$

has the same length as $\lambda(-\bar{w}, \bar{z}) \in T_{(z,w)}S^3$ for all $\lambda \in \mathbb{C}$. We have that

$$dh_{(z,w)}(\lambda(-\bar{w}, \bar{z})) = [\lambda\frac{-|w|^2 - |z|^2}{w^2} : 1] = [\frac{-\lambda}{w^2} : 1]$$

and thus

$$\left\|Dh|_{(z,w)}(\lambda(-\bar{w}, \bar{z}))\right\|_{FS} = \frac{|\frac{-\lambda}{w^2}|^2}{(1 + |\frac{z}{w}|^2)^2} = \frac{\frac{|\lambda|^2}{|w|^4}}{(\frac{|w|^2+|z|^2}{|w|^2})^2} = \frac{\frac{|\lambda|^2}{|w|^4}}{\frac{1}{|w|^4}} = |\lambda|^2 = \|\lambda(-\bar{w}, \bar{z})\|.$$

Next, we introduce foliations. They are a generalization of Riemannian submersions, in the sense that they capture the decomposition of a manifold into smaller ones (as by the vertical subspaces of a Riemannian submersion), but now, not requiring a well-behaved base space.

Definition 2.16. Subbundles of the tangent bundle of a manifold are called distributions. A distribution $\mathcal{D}$ is integrable if the space of sections of $\mathcal{D}$ is a Lie algebra.

Definition 2.17. Let M be a manifold and $\mathcal{D}$ a distribution. An immersed submanifold N of M is integral if $TN \cong \mathcal{D}|_N$.

Definition 2.18. A foliation of a manifold M with an integral distribution $\mathcal{D}$ is a collection of integral manifolds. A maximal connected integral manifold of a foliation is called a leaf.

Example 2.19 (Continue example 2.15)**.** Recall that the space of vector fields over a manifold is a Lie algebra. Since the vertical tangent space $\mathcal{V}$ of a submersion is closed under the (usual) Lie bracket, the space of sections restricted to $\mathcal{V}$, i.e., the space of vertical vector fields, is a Lie algebra itself. Therefore, it defines an integrable distribution on the total space of the submersion, called the vertical distribution. Thus, in the case of the Hopf fibration, the leaf (corresponding to the vertical distribution) through a point $(z, w) \in S^3$ is the great circle $\{(e^{i\theta}z, e^{\ i\theta}w), \theta \in [0, 2\pi)\}$ and the space of leafs is S^2.

In general, a collection of preimages of a submersion π is a foliation induced by the kernel π_*, called the vertical distribution. Conversely, the leaves of a foliation are locally given by preimages of a submersion. So, for a foliation to come from a submersion, the space of leaves must be a well-behaved manifold.

Definition 2.20. A foliation on a Riemannian manifold M is a Riemannian foliation if

1. M is the partition into the maximal disjoint integral manifolds, the leaves, of an integrable distribution.

2. Every geodesic that is normal to a leaf is normal to every other leaf it meets, i.e., the leaves are equidistant.

We call leaves regular if they have maximal dimension under all leaves of a distribution and singular otherwise. The codimension of a singular Riemannian foliation (i.e., a Riemannian foliation with singular leaves) is the codimension of a regular leaf.

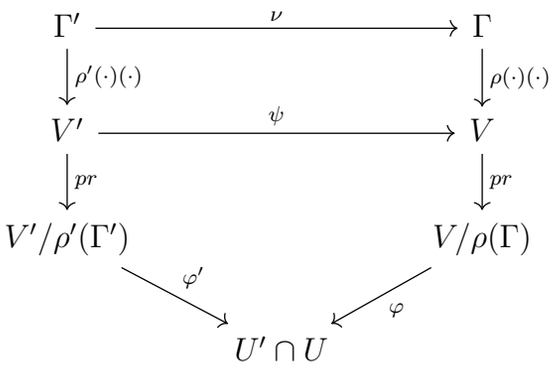
Example 2.21 (Continue 2.3)**.** S^2 has a singular Riemannian foliation. The orbits of the $SO(2)$ action give rise to a partition of S^2 into integral manifolds corresponding to the vertical distribution. The north and south poles give the singular leaves, and the regular leaves are the orbits of principal type. In general, every cohomogeneity one manifold is a singular Riemannian foliation of codimension one. The converse, however, does not hold.

2.3 Orbifolds and Seifert Fibrations

This section follows [9]. Orbifolds are topological spaces defined by charts. Those charts allow one to describe local singularities by a finite group.

Definition 2.22. Let G be a compact Lie group that acts on $\mathbb{R}^n$. Assume the action is effective, i.e., no other element than the identity fixes every other element. A topological space B is an orbifold if

1. It admits a countable basis.
2. B admits orbifold charts: $\forall b \in B \, \exists U \subseteq B$ neighborhood of $b, V \subseteq \mathbb{R}^n$ open and $\rho : \Gamma \hookrightarrow G \to Gl_n(\mathbb{R})$ a finite group such that V is invariant under the effective action of Γ and such that there is a homeomorphism
$$\varphi : V/\rho(\Gamma) \to U, \text{ with } \varphi(0) = b.$$
We call Γ the isotropy group and ρ the isotropy representation at b. Let $\tilde{\varphi}$ denote the composition $\varphi \circ pr : V \to U$.
3. There exist coordinate change maps ψ and intertwining homomorphism ν: for $b \in U \subseteq B, b' \in U' \subseteq B$ and φ, φ' their orbifold charts, there exists a smooth, open embedding $\psi : \tilde{\varphi'}^{-1}((U \cap U')) \to V$ and a group homomorphism $\nu : \Gamma' \to \Gamma$ such that



$$\psi \circ \rho'(\gamma') = \rho(\nu(\gamma')) \circ \psi \quad \forall \gamma' \in \Gamma', \tag{2.2}$$
$$\tilde{\varphi} \circ \psi(v') = \tilde{\varphi}'(v') \quad \forall v' \in \tilde{\varphi'}^{-1}(U \cap U') \tag{2.3}$$

Points for which the isotropy group is not trivial are called singular. We call B effective if the isotropy groups act effectively. We assume in the following that all orbifolds are effective.

Remark 2.23. Coordinate changes are not unique. Given ψ, ν as above, one can define new coordinate changes and intertwining homomorphisms for each $\gamma \in \Gamma$. Define the new coordinate change to be $\rho(\gamma) \circ \psi$ then the new intertwining homomorphism is given by conjugation $\gamma \cdot \nu(\cdot) := \gamma \nu(\cdot) \gamma^{-1}$.

Proof.

$$\begin{aligned} \rho(\gamma \cdot (\nu(\gamma')) \circ (\rho(\gamma) \circ \psi) &= \rho(\gamma)\rho(\nu(\gamma'))\rho(\gamma)^{-1} \circ \rho(\gamma) \circ \psi \\ &= \rho(\gamma)\rho(\nu(\gamma')) \circ \psi \\ &\overset{(2.2)}{=} \rho(\gamma) \circ \psi \circ \rho'(\gamma') \\ \tilde{\varphi} \circ (\rho(\gamma) \circ \psi(v')) = \tilde{\varphi} \circ \psi(v') &\overset{(2.3)}{=} \tilde{\varphi}'(v') \end{aligned}$$

□

Example 2.24. 1. Each manifold is an orbifold with all isotropy groups being the identity.

2. The quotient of a smooth manifold by an almost free action of a compact Lie group is an orbifold with isotropy groups the ones of the action.

3. Consider $\mathbb{R}^n$ with the action of $\mathbb{Z}_2$ given by $(x, y) \to (-x, -y)$. Then $\mathbb{R}^n/\mathbb{Z}_2$ is the cone over the sphere S^{n-1} with singularity the origin. Figure 2.5 illustrates the two-dimensional case. The blue point is the singularity, and the red arrows demonstrate the action of $\mathbb{Z}_2$.

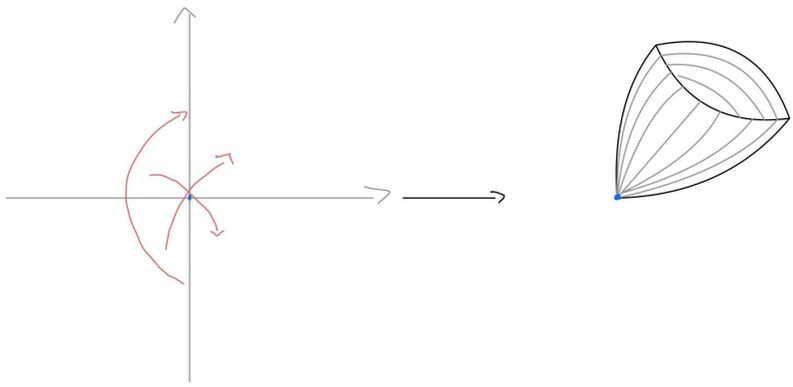


Figure 2.5: $\mathbb{R}^2/\mathbb{Z}_2$

To make sense of fiber bundles over an orbifold, one needs a definition that ensures compatibility of the local trivializations with the singularities.

Definition 2.25. Let B be a G-orbifold and F a manifold. An orbi-bundle with fiber F is a topological space M with a map $\pi : M \to B$ such that

1. There exist local trivializations compatible with the orbifold charts: $\forall b \in B$ there exist Γ, ρ, V, U, ψ as in 2.22, a smooth action $\hat{\rho}$ of Γ on $V \times F$ with $pr : V \times F \to F$ being $\hat{\rho}$ equivariant and a homeomorphism

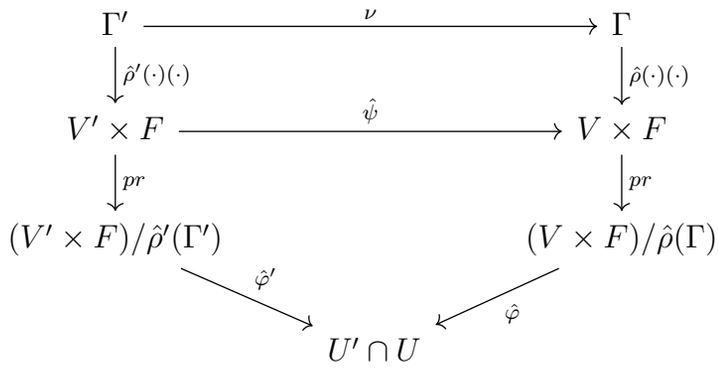
$$\hat{\varphi} : (V \times F)/\hat{\rho} \to \pi^{-1}(U)$$

such that the following diagram commutes.

$$\begin{array}{ccccc} V \times F & \xrightarrow{pr} & (V \times F)/\hat{\rho}(\Gamma) & \xrightarrow{\hat{\varphi}} & \pi^{-1}(U) \\ \downarrow{\scriptstyle pr} & & & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{pr} & V/\rho(\Gamma) & \xrightarrow{\varphi} & U \end{array}$$

2. There exist transition maps compatible with the intertwining homomorphism: Let $\varphi, \varphi', \Gamma, \Gamma', \rho, \rho', \psi, \nu$ as in 2.22 and $\hat{\varphi}, \hat{\varphi}', \hat{\rho}, \hat{\rho}'$ as above. Then there exists a smooth, open embedding $\hat{\psi} : \hat{\varphi}'^{-1}((U \cap U')) \to V \times F$ such that

$$\begin{aligned} \hat{\psi} \circ \hat{\rho}'(\gamma') &= \hat{\rho}(\nu(\gamma')) \circ \hat{\psi} \quad \forall \gamma' \in \Gamma', \\ (\hat{\varphi} \circ pr) \circ \hat{\psi}(v', f) &= (\hat{\varphi}' \circ pr)(v', f) \quad \forall (v', f) \in \hat{\varphi}'^{-1}(U \cap U') \end{aligned}$$



As in the case of usual fiber bundles (cf. A), orbi-bundles appear as associated bundles to principal G- orbi-bundles. For a principal $G-$orbi-bundle, the fiber is a Lie group G and the maps $\hat{\rho}, \hat{\psi}$ commute with the action of G on itself.
The definition of a tangent bundle can be extended to an orbifold. One defines the tangent space at a regular point similarly to the definition on a manifold, but for a singular point, to be modulo the induced action by Γ.

Definition 2.26. Let M, π be an orbi-bundle. π is called Seifert fibration if the $\hat{\rho}$ actions are free. Then M has a smooth manifold structure induced by $\hat{\varphi}$ and the structure of F.

One can view a Seifert fibration π with compact fibers also as a Riemannian foliation with compact leaves. A foliation is given by the vertical distribution (i.e., by the kernel of $d\pi$). Conversely, if $\mathcal{F}$ is a foliation of a manifold M with $B := M/\mathcal{F}$ the leaf space, then $\pi : M \to B$ is a Seifert fibration. This can be seen by choosing orbifold charts induced by the normal exponential map from a normal disc to a leaf L and isotropy groups being the fundamental group of the leaf modulo the kernel of $\pi_1(L, l) \to O(N_l), l \in L, N_l$ normal bundle in M. Note that the normal exponential map is an injective local diffeomorphism for a suitable radius due to compactness. See Remark 1.3 [9] for details.

When considering a Seifert fibration $\pi : M \to B$ with a Riemannian metric on M, we are also able to define horizontal and vertical sub-bundles as well as Riemannian submersions. Since the leaves of a Seifert fibration are manifolds, we can define the vertical sub-bundle $\mathcal{V}$ of TM as the vectors along the leaves and the horizontal sub-bundle $\mathcal{H}$ as the vectors orthogonal to that. If g_B is now a metric on B and $d\pi_p|_{\mathcal{H}} : \mathcal{H} \to T_{\pi(p)}B$ is an isometry we call π a Riemannian submersion (cf. 2.11).

Remark 2.27. Let M be a manifold, G be a compact Lie group, and $H \subset G$ a closed subgroup such that the action of G on M is almost free (i.e., the isotropy groups are finite) and that of H is free. Then $\pi : M/H \to M/G$ is a Seifert fibration with fiber G/H. This can be deduced from the discussion above and from the fact that M/H is foliated by the orbits G/H.

Next, we consider the inertia orbifold ΛB corresponding to an orbifold B. It somehow unfolds the original orbifold B by adding, for each singularity, additional points that capture the symmetric behavior of the isotropy group around the singularity. Those points are called twisted sectors.

Definition 2.28. The intertia orbifold ΛB of an orbifold B is the orbifold defined by points $(b, [\gamma]), b \in B, [\gamma]$ conjugacy class of γ in Γ isotropy group of b.

To see that this indeed defines an orbifold, consider the charts around $(b, [\gamma])$ given by

$$\psi_{[\gamma]} : V^\gamma \backslash \rho(Z_\Gamma(\gamma)) \to \tilde{\psi}(V^\gamma) \times \{[\gamma]\} \tag{2.4}$$

where $\tilde{\psi} : V \backslash \rho(\Gamma) \to U$ was an orbifold chart of B around b, $V^\gamma \subseteq V$ fixed point set of γ and $Z_\Gamma(\gamma)$ the centralizer. Note that the inertia orbifold is no longer necessarily effective. The ineffective kernel corresponding to $(b, [\gamma])$ is defined by

$$m(\gamma) := \#\{\psi \in Z_\Gamma(\gamma) | \rho(\psi)|_{V^\gamma} = id_{V^\gamma}\} \tag{2.5}$$

and is called multiplicity. Note further that $B \subset \Lambda B$ since $(b, [id]) \in \Lambda B \; \forall b \in B$. There is a double cover of the inertia orbifold, which plays a crucial role when later defining the $\hat{A}$- and $L-$ form for an orbifold (see 5.14 and 5.13). For that, consider the normal bundle N_γ of $V^\gamma, \gamma \in \Gamma$. γ acts trivially on V^γ but effectively on V, thus there must be some associated $\gamma' \in SO(rk(N_\gamma))$ acting effectively on the fibers of the normal bundle. Now the lift of γ under the double cover $Spin(rk(N_\gamma)) \to SO(rk(N_\gamma))$ is uniquely determined up to sign (and uniquely if a spin structure is given). Note that $\tilde{\gamma}$ and $-\tilde{\gamma}$ do not lie in the same conjugacy class. Then a double cover of ΛB is given by

$$\tilde{\Lambda B} = \{(b, [\tilde{\gamma}]) | \tilde{\gamma} \text{ lift of } \gamma\} \to \Lambda B. \tag{2.6}$$

$\tilde{\Lambda} B$ is also an orbifold with charts defined analogously to (2.4).

Chapter 3

Spin Geometry

This chapter gives an overview of the most important definitions and results from Spin Geometry, following [26] and [42]. For a short review on covering spaces, see B.3 and on vector bundles, see A. All algebras are assumed to be associative and have a unit element. Furthermore, all fields are supposed to be commutative with characteristic unequal to two. Let k be such a field and V a k-vector space. A quadratic form $q : V \to k$ on V is a map such that $q(\alpha v) = \alpha^2 v \ \forall \alpha \in k, v \in V$ and $q(v, w) := \frac{1}{2}(q(v + w) - q(v) - q(w))$ is bilinear $\forall v, w \in V$.
The overall aim of this chapter is a general understanding of spin structures as well as of the Dirac operator and the Index Theorems 3.42, 3.43 and 3.62. Since we can find coboundaries of homotopy spheres that admit spin structures (cf. 3.25), we are in particular able to apply the Index Theorems to them, which will be the foundation for the Eells-Kuiper invariant in 5.3.1 and 5.3.2. We begin in Section 3.1 with the purely algebraic foundation: Clifford Algebras. This is the algebraic setting that allows one to multiply vectors in a way that respects a quadratic form. Clifford Algebras are matrix algebras and are constructed from a tensor algebra over a vector space V (over some field k) and a quadratic form q. They capture several well-known algebras, such as the Complex numbers and the Quaternions. As we will see, they are instrumental in gathering additional information (beyond $SO(V, q)$ or $O(V, q)$) about rotations and reflections in a vector space, leading to the definition of the groups Spin and Pin. In Section 3.2, we move from the group Spin to defining spin manifolds. From a discussion in Section 3.1 about describing rotations with Spin over $\mathbb{R}^n$ and with respect to the standard quadratic form, we define spin structures. Such structures considered on the tangent bundle of manifolds extend the definition of an orientation of a manifold in the doubled (or quadruple if $\sqrt{-1} \in k$) information sense, cf. 3.11. Stiefel-Whitney classes allow one to determine whether orientations and spin structures exist. With this spin structure in place, Section 3.3 is dedicated to constructing Clifford and spinor bundles. Those are bundles over an n-dimensional manifold M and with fiber Clifford algebras (over $\mathbb{R}^n \cong T_pM \ \forall p \in M$, with the standard quadratic form) or modules of them respectively. By the definition, elements of the spinor bundle, called spinors, can be multiplied by Clifford algebra elements, especially by tangent vectors. Whereas tangent vectors are mapped to themselves when rotating about 360°, spinors are flipped to their negatives. In particular, spinor bundles to tangent bundles over

exotic spheres exist. Finally, in Section 3.4, we introduce the central analytical tool, the Dirac operator. After showing that there exist covariant derivatives on Clifford and thus on spinor bundles in 3.4.1, we will define Dirac operators in terms of such derivations in 3.4.2 (or more generally on Dirac bundles). Note that different Clifford algebra modules give rise to distinct spinor bundles and, consequently, distinct Dirac operators on the same manifold. In 3.4.3 we discuss the Index Theorems, which surprisingly relate a Dirac operator of a spin manifold with topological invariants of the manifold, the $\hat{A}$- and L-genus. Those are polynomials in Pontryagin numbers (cf. C.8). It further turns out that the $L-$genus of an oriented even-dimensional manifold is its signature. To provide a better understanding and to prove the main ideas, we present an analogy to Hodge theory, some foundations in K-theory, and the definition of the Chern character. As a last result, we state the Index Theorems for manifolds with boundary and discuss the equivariant case.

3.1 Clifford Algebras and the Groups Pin, Spin

Definition 3.1. Let $T(V) := \sum_{r=0}^{\infty} \bigotimes^r V$ be the tensor algebra of V and $J_q(V)$ the ideal generated by elements $v \otimes v + q(v)\mathbf{1}, v \in V, q$ quadratic form. Then the Clifford algebra associated to V and q is defined as

$$Cl(V,q) := T(V)/J_q(V).$$

We denote the so-called Clifford multiplication of two elements v, w in $Cl(V,q)$ by $v \cdot w$ or simply vw.

Remark 3.2. V embeds into $Cl(V,q)$ since an element $\varphi \in J_q(V) \cap V$ is on the one hand of the form $\sum_{i=0}^{\infty} a_i \otimes ((v_i \otimes v_i) + q(v_i)) \otimes b_i,\ a_i, b_i$ of pure degree, but on the other hand $\sum_{i=0}^{\infty} a_i \otimes (v_i \otimes v_i) \otimes b_i = 0$ because of φ being an element of $V = \bigotimes^1 V$. So, either one of a_i, b_i is zero or $v_i = 0$ for all i. This implies $\sum_{i=0, a_i, b_i \neq 0}^{\infty} a_i \otimes q(v_i) \otimes b_i = 0$ and thus $\varphi = 0$.

Remark 3.3. $Cl(V,q)$ is as a vector space isomorphic to $\bigwedge^* V$ and as an algebra, if $q = 0$. For that, note, that for orthogonal vectors $e_1, e_2 \in Cl(V,q)$ with respect to the bilinear form induced by q, we have $e_1 e_2 = -e_2 e_1$ in $Cl(q,V)$, since

$$\begin{aligned} 0 &= -2q(e_1, e_2) \bmod J_q(V) \\ &= -q(e_1 + e_2) + q(e_1) + q(e_2) \bmod J_q(V) \\ &= (e_1 + e_2) \otimes (e_1 + e_2) - e_1 \otimes e_1 - e_2 \otimes e_2 \bmod J_q(V) \\ &= e_1 \otimes e_2 + e_2 \otimes e_1 \bmod J_q(V). \end{aligned}$$

One defines an isomorphism of the vector spaces by sending a wedge product $(v \wedge w)$ of two vectors $v, w \in V$ to $\frac{1}{2}(vw - wv)$ and, on the other hand, vw to $v \wedge w - q(v,w)$.

Example 3.4. Let $k = \mathbb{R}, V = \mathbb{R}^n$ and q be the quadratic form induced by the standard Euclidean scalar product, i.e., $q(v) = |v|^2, q(v,w) = \langle v, w \rangle$. $Cl(\mathbb{R}^n, q)$ then has basis

$$\{1, e_1, ..., e_n\} \cup \{e_{i_1} \otimes ... \otimes e_{i_k} | i_1 < i_2 < ... < i_k, 2 \leq k \leq n\},$$

where $\{e_1, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$. Especially $dim_{\mathbb{R}}(Cl(\mathbb{R}^n, q)) = 2^n$. If $\{f_1, ..., f_{2^n}\}$ is the standard basis of $\mathbb{R}^{2^n}$ we can define a vector space isomorphism by

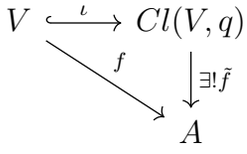
$$Cl(\mathbb{R}^n, q) \to \mathbb{R}^{2^n}$$
$$\left\{ \begin{array}{l} 1 \\ e_i \\ \{e_{i_1} \otimes ... \otimes e_{i_k} | i_1 < i_2 < ... < i_k, 2 \leq k \leq n\} \end{array} \right\} \to \left\{ \begin{array}{l} f_1 \\ f_{i+1} \\ \{f_{n+2}, ..., f_{2^n}\} \end{array} \right\}$$

Therefore we can also introduce a topology on $Cl(\mathbb{R}^n, q)$ which makes $Cl(\mathbb{R}^n, q)$ naturally homeomorphic to $\mathbb{R}^{2^n}$. Even as $\mathbb{R}-$algebras we obtain, $Cl(\mathbb{R}, q) \cong \mathbb{C}$, $Cl(\mathbb{R}^2, q) \cong \mathbb{H}$, which can be easily checked: If e is the basis vector of $\mathbb{R}$, then we have that $e^2 = -1$ in $Cl(\mathbb{R}, q)$ and thus can be interpreted as i. In the second case with e_1, e_2 basis vectors of $\mathbb{R}^2$ we have in $Cl(\mathbb{R}^2, q)$ the relations

$$e_1^2 = -1, e_2^2 = -1, e_1 \otimes e_2 = -2q(e_1, e_2) - e_2 \otimes e_1 = -e_2 \otimes e_1$$

so that we can interpret e_1 as i, e_2 as j and $e_1 e_2$ as k. Analogously one can conclude $Cl(\mathbb{R}^3, q) \cong \mathbb{H} \oplus \mathbb{H}, Cl(\mathbb{R}^4, q) \cong M_2(\mathbb{H}), Cl(\mathbb{R}^5, q) \cong M_4(\mathbb{C}), Cl(\mathbb{R}^6, q) \cong M_8(\mathbb{R})$, $Cl(\mathbb{R}^7, q) \cong M_8(\mathbb{R}) \otimes M_8(\mathbb{R}), Cl(\mathbb{R}^8, q) \cong M_{16}(\mathbb{R})$, where $M_n(\mathbb{R})$ denote the $n \times n$ matrix algebras over $\mathbb{R}$. It can be further shown that $Cl(\mathbb{R}^{n+8}, q) \cong Cl(\mathbb{R}^n, q) \otimes Cl(\mathbb{R}^2, q)$ and $Cl(\mathbb{C}^n, q') \cong Cl(\mathbb{R}^n, q) \otimes \mathbb{C}$, where q' is the quadratic form defined by the complex standard inner product (cf. pp. 24 [26] for the proofs).

The Clifford algebra $Cl(V, q)$ over k is initial in the category of k-algebras that admit a linear map f from the k-vector space V to itself satisfying $f(v)f(v) = q(v)$. That means such a map f factorizes over the inclusion of that vector space into $Cl(V, q)$. The extension of f to the Clifford algebra is easy to see by defining it as the result of applying f to each component of a tensor product. The uniqueness follows from defining an isomorphism of another algebra to the Clifford algebra via the identification of the embedded V.



Given two $k-$ vector spaces with quadratic forms $(V, q), (\tilde{V}, \tilde{q})$ and a $k-$ linear map $f : V \to \tilde{V}$ that preserves the quadratic form, we can apply the universal property to $V \to \tilde{V} \hookrightarrow Cl(\tilde{V}, \tilde{q})$ and get an homomorphism $\bar{f} : Cl(V, q) \to Cl(\tilde{V}, \tilde{q})$. The uniqueness part ensures that compositions of maps are preserved under the bar construction, and thus we get the following functorial relation.

$$\{k - \text{Vector spaces with an quadratic form }\} \to \{\text{Clifford algebras over } k\} \quad (3.1)$$
$$(V, q) \to Cl(V, q)$$
$$\{f : V \to \tilde{V} \text{ with } f^* \tilde{q} = q\} \to \{\bar{f} : Cl(V, q) \to Cl(\tilde{V}, \tilde{q})\}$$

As an immediate consequence, we get

$$O(V, q) := \{f \in Gl(V), f \text{ preserves } q\} \subset Aut(Cl(V, q)). \quad (3.2)$$

Definition 3.5. Consider $\alpha : V \to V, \alpha(v) = -v$. Under the identification 3.2 we get a decomposition

$$Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$$

of the Clifford algebra into the plus and minus one eigenspaces of α^2, called the even and odd parts, respectively. Notice that the names are exactly what they mean, i.e. $v_1 \cdots v_{2k} \in Cl^0(V,q)$ and $v_1 \cdots v_{2k+1} \in Cl^1(V,q)\, \forall v_i \in V, i \in \{1, ..., 2k{+}1\}, k \in \mathbb{N}$. Note, that in contrast to $Cl^0(V,q)$, $Cl^1(V,q)$ is not a subalgebra (odd part times odd part = even part !) but a vector space.

In the following, we want to examine how unit elements of Clifford algebras represent reflections of the underlying vector field. Recall that a representation of a group means a homomorphism from the group to the (linear) transformations of a space. I.e., it describes how the group acts on that space.

Definition 3.6. $Cl^\times(V,q) = \{\varphi \in Cl(V,q) | \exists \varphi^{-1} : \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\}$ is called the multiplicative group of units. We call the map

$$Ad: \ Cl^\times \ (V,q) \ \to \ Aut(Cl(V,q)), Ad_\varphi(x) = \varphi x \varphi^{-1}$$

the adjoint representation of $Cl^\times(V,q)$.

Remark 3.7. Since for $k = \mathbb{R}, \mathbb{C}$ and V finite dimensional, $Cl^\times(V,q)$ is a Lie group, the name of the map Ad is consistent.

Proposition 3.8. *For $v \in V$ with $q(v) \neq 0$ it holds that $Ad_v(V) = V$ and in particular*

$$-Ad_v(w) = w - 2\frac{q(v,w)}{q(v)}v, \forall w \in V. \tag{3.3}$$

Furthermore, Ad preserves the quadratic form, so there is a representation

$$Cl^\times(V,q) \supset P(V,q) := \langle \{v \in V | q(v) \neq 0\} \rangle \xrightarrow{Ad} O(V,q) \xrightarrow{3.2} Aut(Cl(V,q)).$$

Proof.

$$\begin{aligned}
-Ad_v(w) = -vwv^{-1} &= vwvq(v)^{-1} \\
&= ((vw + wv)v - wv^2)q(v)^{-1} \\
&= ((-q(v+w) + q(v) + q(w))v - wv^2)q(v)^{-1} \\
&= (-2q(v,w)v + wq(v))q(v)^{-1} \\
&= w - 2\frac{q(v,w)v}{q(v)} \\
q(Ad_v(w)) &= q(2\frac{q(v,w)v}{q(v)} - w) \\
&= q(2\frac{q(v,w)}{q(v)}v) + 2q(2\frac{q(v,w)v}{q(v)}, -w) + q(-w) \\
&= 4(\frac{q(v,w)v}{q(v)})^2 q(v) - 4\frac{q(v,w)}{q(v)}q(v,w) + q(w) \\
&= q(w)
\end{aligned}$$

□

Equation 3.3 describes a reflection across the hyperplane orthogonal to v. Unfortunately, there is this additional minus sign in front of Ad, which implies that Ad is orientation preserving if n is odd, and reversing if n is even. To get rid of this, one defines the twisted adjoint representation via the extended homomorphism α from Definition 3.5:

$$\tilde{Ad} : Cl^{\times}(V,q) \to Aut(Cl(V,q)), \tilde{Ad}_{\varphi}(x) := \alpha(\varphi)x\varphi^{-1}.$$

Then one has

$$\tilde{Ad}_v(w) = w - 2\frac{q(v,w)}{q(v)}v \quad \forall v, w \in V$$

and the representations

$$P(V,q) \xrightarrow{\tilde{Ad}} O(V,q) \xrightarrow{3.2} Aut(Cl(V,q)) \tag{3.4}$$

$$SP(V,q) := \{v_1 \cdots v_r \in P(V,q) | q(v_i) \neq 0, \text{ r even } \} = P(V,q) \cap Cl^0(V,q)$$

$$\xrightarrow{\tilde{Ad}} SO(V,q) := \{\gamma \in O(V,q) : det(\gamma) = 1\} \xrightarrow{3.2} Aut(Cl(V,q)). \tag{3.5}$$

Elements in the orthogonal group always can be written as a product (in the special orthogonal group as an even number) of reflections (cf. Theorem 2.7 [26]), so we have surjectivity of the $\tilde{Ad}$ homomorphisms in 3.4 and 3.5. Our goal is to find the smallest subgroups of $P(V,q)$ which map onto $O(V,q)$ and $SO(V,q)$. To do so, we define subgroups of $P(V,q)$ and $SP(V,q)$ by restricting them to unit vectors.

Definition 3.9. Pin and Spin are subgroups of $P(V,q)$ and are defined as

$$\text{Pin}(V,q) := \{v_1 \cdots v_r \in P(V,q) : q(v_i) = \pm 1 \forall i\} \subset P(V,q)$$
$$\text{Spin}(V,q) := \{v_1 \cdots v_r \in Pin(V,q) : r \text{ is even}\} = \text{Pin}(V,q) \cap Cl^0(V,q) \subset SO(V,q).$$

If $Cl(V,q)$ is a topological space (induced by a topology on V), then $\text{Pin}(V,q)$ and $\text{Spin}(V,q)$ are topological groups. If further for $k = \mathbb{R}, \mathbb{C}$ and V finite dimensional $\text{Pin}(V,q)$ and $\text{Spin}(V,q)$ are as closed subgroups of $Cl^{\times}(V,q)$ Lie groups.

The only thing to check whether $\tilde{Ad}$ maps $\text{Pin}(V,q)$ onto $O(V,q)$ and $\text{Spin}(V,q)$ onto $SO(V,q)$ is, if for every $v \in P(V,q)$ there exists an $\gamma \in k$ such that the equation $\gamma^2 q(v) = q(\gamma v) = \pm 1$ is solvable (i.e. every $v \in V^{\times}$ can be normalized to q length one), since one has that $\tilde{Ad}_{\gamma v}(w) = \tilde{Ad}_v(w)$. We call such fields spin. Especially, this holds for $\mathbb{R}$ and $\mathbb{C}$ with respect to the standard form, but, for example, $\mathbb{Q}$ is not spin.

Example 3.10 (Continue example 3.4)**.** Every rotation in $\mathbb{R}^n$ can be expressed as an even number of reflections (which is not unique). We can further describe these reflections via unit vectors $v_1, ..., v_r$ orthogonal to the reflection planes. Figure 3.1 shows an example of a rotation by an angle of $2(\theta - \phi)$ expressed by two reflections in $\mathbb{R}^2$. This rotation thus is given as $\varphi := v_1 v_2$ in $\mathrm{Spin}(\mathbb{R}^2, q)$. The rotation described by $-(v_1 v_2)$ is the same, but is a different element in $\mathrm{Spin}(\mathbb{R}^2, q)$. If the rotation is described by another combination of reflections $\vartheta := v_3 v_4$, then $v_3 v_4 = \pm(v_1 v_2)$ in $\mathrm{Spin}(\mathbb{R}^2, q)$.

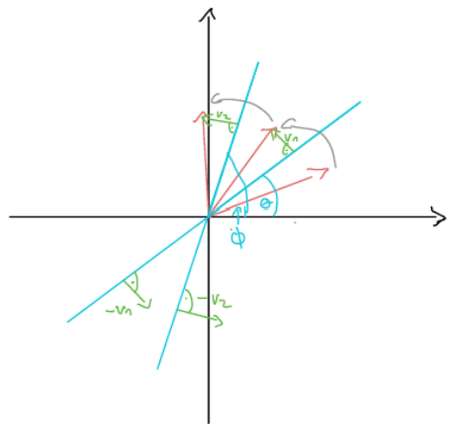


Figure 3.1: Rotation expressed as two reflections in $\mathbb{R}^2$

This is true since it must hold that $\forall x \in Cl(\mathbb{R}^2, q)$

$$\begin{aligned} &\tilde{Ad}_\varphi(x) = \tilde{Ad}_\vartheta(x) \\ \implies &\alpha(\varphi) x \varphi^{-1} = \alpha(\vartheta) x \vartheta^{-1} \\ \implies &x = \alpha(\varphi)^{-1}\alpha(\vartheta) x \vartheta^{-1}\varphi = \varphi^{-1}\vartheta x \vartheta^{-1}\varphi \end{aligned}$$

and the v_i are unit vectors. The observation still holds in the more general setting, as captured in the next Theorem, along with the comments above. For a complete proof, see pp. 16 in [26].

Theorem 3.11. *If V is a finite-dimensional vector space over a spin field k and q is a non-degenerate quadratic form on V, there is a short exact sequence*

$$0 \to F \to Spin(V, q) \xrightarrow{\tilde{Ad}} SO(V, q) \to 1$$

where $F = \{\pm 1\}$ if $\sqrt{-1} \notin k$ or else $F = \{\pm 1, \pm\sqrt{-1}\}$.

If $\sqrt{-1} \notin k$ and V is a topological space then by the theorem $\mathrm{Spin}(V, q)$ is the topological double cover of $SO(V, q)$, and in fact for $n \geq 3$ the universal cover since then $\pi_1(SO(V, q)) \cong \mathbb{Z}_2$ and thus by covering theory $\pi_1(\mathrm{Spin}(V, q))$ is trivial. This further implies that under these assumptions, $\mathrm{Spin}(V, q)$ is, as the universal cover of a manifold, a Lie group. Example 3.10 gives an intuition of how one should think about this. φ and $-\varphi$ describe the same rotation in $SO(V, q)$. However, the distinction between $-\varphi$ and φ gives additional information. Another way of saying this is that rotating by 360° is necessarily represented by a minus sign in $\mathrm{Spin}(\mathbb{R}^2, q)$, which gets eliminated if we rotate once more by 360°. In $SO(\mathbb{R}^2, q)$ both rotations are clearly represented by the identity. In $\mathrm{Spin}(V, q)$, we have more (doubled) information about the rotation.

3.2 Spin Manifolds

For the rest of the chapter, we want to stay in the special case of Example 3.4. We therefore use from now on the notations

$$Cl_n := Cl(\mathbb{R}^n, q), \mathrm{Spin}_n := \mathrm{Spin}(\mathbb{R}^n, q), \mathrm{Pin}_n := \mathrm{Pin}(\mathbb{R}^n, q), \mathbb{C}l_n := Cl_n \otimes \mathbb{C}$$

where q is the standard Euclidean bilinear form. Also, let $\pi : E \to M$ be a smooth, real $n-$dimensional vector bundle over a manifold M (which implies that E is a manifold) with a positive inner product continuously defined on the fibers (called a Riemannian structure). Such a structure always exists, since one can use local trivializations and partitions of unity to construct a global inner product.
We first want to examine how the orientability of vector bundles can be decided via the first Stiefel-Whitney class (cf. C.5 and the following comments for a general definition). A vector bundle is orientable if one can define orientations on each fiber such that they can be mapped orientation-preservingly to one another; i.e., the structure group consists of elements of positive determinant. Thus, if E is orientable, there exist exactly two possible orientations.

Definition 3.12. Let $P_O(E)$ be the bundle of orthonormal frames in E (the term orthonormal makes sense because E has a Riemannian structure). The bundle of orientations of E is $Or(E) := P_O(E)/SO_n$, a two-sheeted covering of M.

For every $x \in M$, the two sheets of the fiber of x in $Or(E)$ correspond to the two possible orientations. Hence, E is orientable if and only if $Or(E)$ is the trivial bundle $M \times \mathbb{Z}_2$. We are therefore interested in principal $\mathbb{Z}_2-$ bundles to decide orientability. Note that equivalence classes of principal $G-$bundles are isomorphic to $H^1(M;G)$ (cf. A.9). There is a bijection between $n-$sheeted coverings of M and subgroups of index n in $\pi_1(M)$ (cf. section B.3). $\pi_1(M)$ being the abelianization of $H_1(M)$ implies

$$H^1(M;G) \cong Hom(H_1(M),G) \cong Hom(\pi_1(M),G). \tag{3.6}$$

In the special case where $G = \mathbb{Z}_2$, one has that the kernel of every non-trivial homomorphism from $\pi_1(M)$ to $\mathbb{Z}_2$ must have index 2 and uniquely determines the homomorphism. Thus the cardinality of $Hom(\pi_1(M),\mathbb{Z}_2)$ is exactly the number of subgroups of index 2 in $\pi_1(M)$. This proves the following Lemma.

Lemma 3.13. *The equivalence classes of two-sheeted covering spaces of M $Cov_2(M)$ (with fiberwise product) are isomorphic to $H^1(M;\mathbb{Z}_2)$.*

Example 3.14. Consider the tangent bundle $p : TS^2 \to S^2$. Let

$$x := \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \in S^2, \theta \in [0,\pi], \phi \in [0,2\pi).$$

Then $v_1(x) := \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}$ and $v_2(x) := \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$ form an orthonormal basis of T_xS^2. Therefore $P_O(TS^2)_x = \{A \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} | A \in O_2(T_xS^2)\}$ and these matrices A are either rotations or reflections. Every rotation is of the form

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in SO_2, \forall\alpha \in [0,2\pi).$$

The reflections are combinations of the matrices $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ which are SO_2 equivalent and have determinant -1. Thus, $Or(TS^2) = S^2 \times \mathbb{Z}_2$ since we have two global sections

$s^{\pm} : S^2 \to Or(TS^2)$ given by $s^+(x) = (v_1(x), v_2(x)), s^-(x) = (-v_1(x), v_2(x)), x \in S^2$,

which describe the orientations of TS^2. It is well known that $H^1(S^2; \mathbb{Z}_2) = 0$ and the Lemma 3.13 implies that there is only one equivalence class of two-sheeted covering spaces of S^2 given by $Or(TS^2)$.

Definition 3.15. The image of $Or(E)$ in $H^1(M; \mathbb{Z}_2)$ under the isomorphism of Lemma 3.13 is called the first Stiefel-Whitney class of E and denoted by $w_1(E)$.

Theorem 3.16. *E is orientable if and only if the first Stiefel-Whitney class $w_1(E)$ is zero. Then the elements in $H^0(M; \mathbb{Z}_2)$ are in one-to-one correspondence with the orientations of E.*

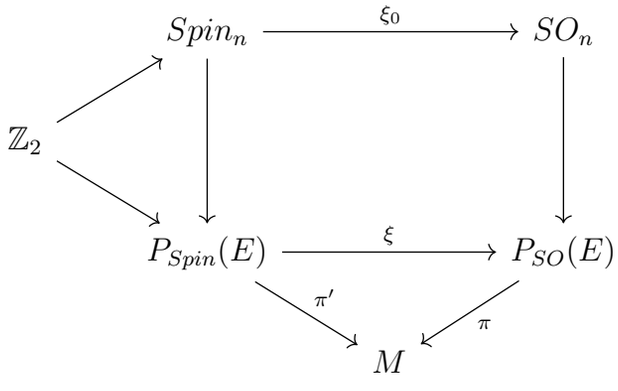
Proof. $w_1(E)$ being the image of $Or(E)$ in $H^1(M; \mathbb{Z}_2)$ implies with (3.6) that $Or(E)$ is mapped to the trivial homomorphism in $Hom(\pi_1(M), \mathbb{Z}_2)$. This is associated with the covering associated with the trivial Deck transformation group (cf. Section 1.3 in [15]) and thus the trivial double cover. Therefore, as mentioned above, E is orientable. Since $H^0(M; \mathbb{Z}_2)$ is the group of functions from the connected components of M to Z_2, its elements represent the possible orientations of E. □

After this discussion about orientations on vector bundles, we now want to define spin structures. Such structures can be defined for orientable vector bundles. So assume that E is orientable. Fixing an orientation is equivalent to choosing a bundle of oriented orthonormal frames $P_{SO}(E)$ or, which is the same, an associated principal SO_n-bundle (see A.12). If it is possible to lift the structure group SO_n to its double cover $Spin_n$ in an equivariant way, we talk about a spin structure.

Definition 3.17. Let $n \geq 3$ and $\xi_0 : \mathrm{Spin}_n \to SO_n$ the universal covering. A spin structure on E is a principal Spin_n-bundle $P_{Spin}(E)$ with a two sheeted covering

$$\xi : P_{Spin}(E) \to P_{SO}(E)$$

where $P_{SO}(E)$ is the associated principal SO_n- bundle and with the property that $\xi(pg) = \xi(p)\xi_0(g) \, \forall p \in P_{Spin}(E)$, $g \in \mathrm{Spin}_n$ (cf. the definition of equivalent bundles A.8).



For $n = 2$ a spin structure is defined analogously but with

$$\xi_0 : SO_2 \to SO_2, \; R_\alpha \to R_{2\alpha}$$

and replacing Spin_2 by SO_2. In the case of $n = 1$ a spin structure is a two sheeted covering of $M \cong P_{SO}(E)$.

Theorem 3.18. *There is an exact sequence*

$$0 \to H^1(M;\mathbb{Z}_2) \xrightarrow{\pi^*} H^1(P_{SO}(E);\mathbb{Z}_2) \xrightarrow{i^*} H^1(SO_n;\mathbb{Z}_2) \xrightarrow{w_E} H^2(M;\mathbb{Z}_2) \tag{3.7}$$

corresponding to the fibration $SO_n \xrightarrow{i} P_{SO}(E) \xrightarrow{\pi} M$.

Proof. Since SO_n is a Lie group, $\pi_1(M)$ acts on SO_n via conjugation: Lift a loop $\gamma \in \pi_1(M)$ horizontally to the unique $\tilde{\gamma}$ on $P_{SO}(E)$ with $\pi(\tilde{\gamma}(0)) = \gamma(0)$(there are multiple choices for $\tilde{\gamma}(0)$ and thus $\tilde{\gamma}$!, cf. A.19). Then one has that $\tilde{\gamma}(0) = g_{\tilde{\gamma}}\tilde{\gamma}(1)$ for some $g_{\tilde{\gamma}} \in SO_n$. Define a homomorphism $\pi_1(M) \to Aut(SO_n), \gamma \to g\tilde{\gamma}$. If one chooses a different starting point for a lift of γ one obtains a conjugate automorphism (since different starting points lead to different horizontal lifts $\tilde{\gamma}, \tilde{\gamma}'$, one would get $\tilde{\gamma}(t) = h\tilde{\gamma}'(t)$ which implies that $g\tilde{\gamma}' = h^{-1}g\tilde{\gamma}h$). So in total, an element in $\pi_1(M)$ determines an inner automorphism of SO_n, which is given by conjugation. This action is homotopic to the identity because of path-connectness of SO_n ($H_t(x) = \gamma(t)x\gamma(t)^{-1}$ is a homotopy between id and Ad_g for a path γ from id to g) and thus trivial on $H^*(SO_n;\mathbb{Z}_2)$. Therefore, with Theorem B.22, there exists a Serre Spectral sequence $\{E_r^{p,q}, d_r\}$ with

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad E_{r+1}^{p,q} = ker(d_r)/im(d_r), E_2^{p,q} \cong H^p(B;H^q(F;G))$$

and stable items $E_\infty^{p,n-p}$. Since SO_n and M are connected one has $E_2^{p,0} \cong H^p(M;\mathbb{Z}_2)$ and $E_2^{0,q} \cong H^q(SO_n;\mathbb{Z}_2)$. Moreover

$$E_3^{0,1} = ker(d_2 : E_2^{0,1} \to E_2^{2,0}) \,/\, im(d_2 : E_2^{-2,2} \to E_2^{0,1})$$

but $E_2^{-2,2} = 0$ so

$$E_\infty^{0,1} \cong ker(d_2 : E_2^{0,1} \to E_2^{2,0}).$$

For the same reason

$$E_\infty^{2,0} \cong E_2^{2,0}/im(d_2 : E_2^{0,1} \to E_2^{2,0})$$

and we get the exact sequence

$$0 \to E_\infty^{0,1} \to H^1(SO_n;\mathbb{Z}_2) \xrightarrow{d_2} H^2(M;\mathbb{Z}_2) \to E_\infty^{2,0} \to 0. \tag{3.8}$$

The Serre Sequence further implies that there is a filtration

$$0 \subset F_1^1 \subset F_0^1 = H^1(P_{SO}(E);\mathbb{Z}_2)$$

with $E_\infty^{0,1} \cong F_0^1/F_1^1$ and $E_\infty^{1,0} = F_1^1$ and thus there is a second exact sequence

$$0 \to E_\infty^{1,0} \to H^1(P_{SO}(E);\mathbb{Z}_2) \to E_\infty^{0,1} \to 0. \tag{3.9}$$

Note, that

$$H^1(M;\mathbb{Z}_2) \cong E_2^{1,0} = ker(d_1 : E_1^{1,0} \to E_1^{2,0})/im(d_1 : E_1^{0,0} \to E_1^{1,0}) \cong E_\infty^{1,0} = F_1^1.$$

So the sequences (3.8) and (3.9) fit together to the sequence

$$0 \to H^1(M;\mathbb{Z}_2) \to H^1(P_{SO}(E);\mathbb{Z}_2) \to H^1(SO_n;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2).$$

□

Example 3.19 (Continue example 3.14)**.** Consider the Hopf Fibration (cf. 2.15)

$$S^1 \to S^3 \to S^2$$

a principal $S^1 \cong SO_2$ bundle over S^2. Indeed, it is a spin structure for S^2:

$$\xi_0 : S^1 \to S^1, \lambda \to \lambda^2 \text{ induces } \xi : S^3 \to P_{SO}(TS^2) \cong S^3 \times_{\xi_0} SO_2, \varphi \to [\varphi, id],$$

which means that S^1 acts via ξ_0 on the fibers of $P_{SO}(TS^2)$. By definition of the associated bundle action

$$\xi(\varphi\lambda) = [\varphi\lambda, id] = [\varphi, \xi_0(\lambda)] = \xi(\varphi)\xi_0(\lambda).$$

It holds that $\pi_1(S^3) = 0$ and because of Proposition B.17

$$\pi_1(P_{SO}(TS^2))/\xi_*(\pi_1(S^3)) \;\cong\; \mathbb{Z}_2$$

we get that $\pi_1(P_{SO}(TS^2) \cong \mathbb{Z}_2$. Thus, $H^1(P_{SO}(TS^2); \mathbb{Z}_2) = 0$ and Lemma 3.13 implies that there is exactly one two-sheeted covering of $P_{SO}(TS^2)$ and thus the only spin structure on S^2 is the Hopf Fibration.

Definition 3.20. Let $n \geq 2$ and recall that $H^1(SO_n; \mathbb{Z}_2) \cong \mathbb{Z}_2$. The image of the generator g in $H^1(SO_n; \mathbb{Z}_2)$ under w_E is denoted by $w_2(E)$ and called the second Stiefel-Whitney class of E.

Analogously to Theorem 3.16, we get the following result.

Theorem 3.21. *There exists a spin structure on E if and only if $w_2(E) = 0$. Then the elements in $H^1(M; \mathbb{Z}_2)$ are in one-to-one correspondence with the distinct structures.*

Proof. If one has a two sheeted covering $\xi : P \to P_{SO}(E)$ which is non trivial on the fibers of π (i.e. the covering restricted to the fibers is the only non-trivial double covering $\xi_0 : \mathrm{Spin}_n \to SO_n$, i.e. Diagram 3.2 with replacing $P_{Spin}(E)$ by P commutes), then one can make it into a principal Spin-bundle. This is true since one can lift for all $g \in SO_n$ the map $f_g : P_{SO}(E) \to P_{SO}(E), x \to gx$ with the lifting criterion B.16 to a map $\tilde{f}_g : P_{SO}(E) \to P$, i.e. we have that $f_g = \xi \circ \tilde{f}_g$. In order to obtain a Spin action on P define for $\tilde{g} \in \mathrm{Spin}_n$ the map $F_{\tilde{g}} : P \to P, x \to \tilde{f}_{\xi_0(\tilde{g})} \circ \xi$. This action is compatible in the sense that the Diagram 3.2 commutes.

$$\begin{array}{ccc} P & \xrightarrow{F_{\tilde{g}}} & P \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ P_{SO}(E) & \xrightarrow{f_{\xi_0(\tilde{g})}} & P_{SO}(E) \end{array}$$

Figure 3.2: Lifted spin action

Therefore, we have seen that spin structures on E are in one-to-one correspondence with two-sheeted coverings of $P_{SO}(E)$, i.e., with Lemma 3.13 with elements in $H^1(P_{SO}(E); \mathbb{Z}_2)$, which are non-trivial on the fibers. Now, together with Theorem 3.18 we have

$$\begin{aligned} w_2(E) = 0 &\Leftrightarrow ker(w_E) = H^1(SO_n; \mathbb{Z}_2) \\ &\Leftrightarrow im(i^*) = H^1(SO_n; \mathbb{Z}_2) \\ &\Leftrightarrow \text{ There exists a spin structure on } E \end{aligned}$$

and $H^1(SO_n;\mathbb{Z}_2) \cong \mathbb{Z}_2 \cong H^1(P_{SO}(E);\mathbb{Z}_2)/H^1(M;\mathbb{Z}_2)$. Thus, elements in $H^1(M;\mathbb{Z}_2)$ are in one-to-one correspondence with two-sheeted coverings of $P_{SO}(E)$ that are non-trivial on the fibers and therefore with spin structures on E (half of the elements in $H^1(P_{SO}(E);\mathbb{Z}_2)$ correspond to the trivial coverings on the fibers, i.e. where $\xi_0 : SO_n \sqcup SO_n \to SO_n$).

□

Definition 3.22. An oriented manifold M with a spin structure on the tangent bundle is called a spin manifold. The Stiefel-Whitney classes of M are defined to be the classes of the tangent bundle. Two spin manifolds are differentiably equivalent if there exists an orientation- and spin-structure-preserving diffeomorphism between them.

Example 3.23.

1. As seen in Example 3.14 S^2 is spin and in fact every $n-$sphere is. Therefore, every exotic sphere is spin (the Stiefel-Whitney classes are topological invariants).

2. Parallizable, oriented manifolds are spin. An n-dimensional, $n \geq 2$, manifold M is parallizable if its tangent bundle is trivial. Therefore, $P_{SO}(TM)$ is trivial (it has a global section) and thus, with the Künneth formula (B.6)

$$\begin{aligned} H^1(P_{SO}(TM);\mathbb{Z}_2) \cong & (H^1(M;\mathbb{Z}_2) \otimes H^0(SO_n;\mathbb{Z}_2)) \\ & \oplus (H^0(M;\mathbb{Z}_2) \otimes H^1(SO_n;\mathbb{Z}_2)) \\ \cong & (H^1(M;\mathbb{Z}_2) \otimes \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \\ \cong & H^1(M;\mathbb{Z}_2) \oplus \mathbb{Z}_2 \end{aligned}$$

   Putting this into the exact sequence (3.7), one can conclude that i^* must be surjective and consequently the second Stiefel-Whitney class of M must be zero.

3. If M, M_1, M_2 are manifolds with $M = M_1 \times M_2$, then spin structures on any two of the three determine a spin structure on the third (Proposition 2.15 [26]). Especially, a submanifold M' of a spin manifold M with spin structure on its normal bundle is a spin manifold ($TM = N(M') \bigoplus TM'$).

Last, we discuss spin cobordism. Recall that the group of exotic spheres (the group operation is the connected sum) consists of h-cobordism classes of homotopy spheres (cf. Theorem 1.8). Two oriented manifolds M, N are said to be h-cobordant if there exists an oriented manifold W with boundary $\partial W = M \sqcup N$ and $M \hookrightarrow W, N \hookrightarrow W$ are homotopy equivalences (this is an equivalence relation).

Definition 3.24. Two spin manifolds are spin cobordant if their disjoint union bounds a spin manifold. The free abelian group generated by the equivalence classes (differentiable equivalence) of $n-$dimensional, compact, spin manifolds modulo the subgroup generated by the sum of elements whose disjoint unions are spin cobordant to zero (i.e., bounds a spin manifold) is denoted by Ω_n^{Spin}.

Remark 3.25. For the definition of the Eells-Kuiper invariant in section 5.3.1, it will be of interest to consider coboundaries of exotic seven-spheres (i.e., eight-dimensional

manifolds with exotic spheres as their boundary). Primarily, we use the argument that any exotic seven-sphere bounds an eight-dimensional spin manifold, which is because of $\Omega_7^{Spin} = 0$ (cf. p. 92 [26]).

3.3 Spinor Bundles

Let E be an oriented Riemannian vector bundle. Clifford bundles are the associated bundles to $P_{SO}(E)$ with fiber Cl_n. They are therefore bundles of algebras.

Definition 3.26. Let $p_n : SO_n \to SO(\mathbb{R}^n)$ be the representation of SO_n as orientation preserving orthogonal transformations of $\mathbb{R}^n$. Of course, each such transformation induces one of Cl_n (the quadratic form is preserved). Therefore, we have a representation

$$Cl(p_n) : SO_n \to Aut(Cl_n).$$

The real Clifford bundle corresponding to E is defined to be

$$Cl(E) := P_{SO}(E) \times_{Cl(p_n)} Cl_n \text{ and for the complex case } \mathbb{C}l(E) := Cl(E) \otimes \mathbb{C}.$$

Extending the Automorphism $\alpha : \mathbb{R}^n \to \mathbb{R}^n, v \to -v$ from Definition 3.5 to $\tilde{\alpha} : Cl(E) \to Cl(E)$ gives again a splitting $Cl(E) = Cl^0(E) \oplus Cl^1(E)$ into the $\pm 1-$ eigenbundles of $\tilde{\alpha}$ (i.e. the fibers of $Cl^0(E)$ and $Cl^1(E)$ at x are the $+1$ and -1 eigenspace of $\tilde{\alpha}_x : Cl(E)_x \to Cl(E)_x$) respectively. Further, there exists another splitting $Cl(E) = Cl^+(E) \oplus Cl^-(E)$ given by the volume element

$$\omega = e_1 \cdots e_n \in Cl_n$$

where $\{e_1, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$. Since $e_i e_j = -e_j e_i, i \neq j$, we obtain

$$\omega^2 := (-1)^{\frac{n(n-1)}{2}} e_1 \cdots e_n e_n \cdots e_1 = (-1)^{\frac{n(n-1)}{2}} (-1)^n$$

and thus $\omega^2 = 1$ if $n \equiv 0, 3 \bmod 4$. Since this volume element extends globally we get in those dimension cases a splitting into the plus and minus one eigenspaces of the volume element, denoted by $Cl^+(E)$ and $Cl^-(E)$ respectively. Analogously, one obtains a splitting of $\mathbb{C}l(E)$ whenever $n = 2m$ is even but by the complex volume element

$$\omega_{\mathbb{C}} := i^m e_1 \cdots e_{2m}.$$

The dimension being even ensures that $\omega_{\mathbb{C}}$ squares to 1.

Recall from Remark 3.3 that $Cl_n \cong \bigwedge^* \mathbb{R}^n$ as vector spaces. As a direct consequence, we get the following.

Proposition 3.27. *$Cl(E)$ and $\bigwedge^*(E) = P_{SO}(E) \times_{\bigwedge^* q_n} (\bigwedge^* \mathbb{R}^n)$ where q_n is given by sending $A \in GL(\mathbb{R}^n)$ to $(x \to Ax)$, are isomorphic as vector bundles. Further one has under the same isomorphism $\bigwedge^{even} \cong Cl^0(E), \bigwedge^{odd} \cong Cl^1(E)$.*

Now that we have a good understanding of Cl_n and $Cl(E)$, it is natural to ask for Clifford modules that will yield the definition of spinor bundles. For that, we need to make sense of multiplying an element from Cl_n with elements from a vector space. The following definition applies to general Clifford algebras and vector spaces over fields, but we want to stay in our special setting of Cl_n.

Definition 3.28. Let k be a field containing $\mathbb{R}$ and W a finite-dimensional k-vector space. A $k-$representation of Cl_n is a $\mathbb{R}-$algebra homomorphism

$$\psi : Cl_n \to Hom_k(W, W).$$

W is called Cl_n-module over k. One defines the Clifford multiplication (i.e., the module multiplication) to be

$$\varphi w := \psi(\varphi)(w) \text{ for } \varphi \in Cl_n, w \in W.$$

Analogously one defines a $\mathbb{C}l_n$-module $W_{\mathbb{C}}$ for a field containing $\mathbb{C}$.

Definition 3.29. Let E be oriented with a spin structure $\xi : P_{Spin}(E) \to P_{SO}(E)$ and W a left Cl_n module. Let further be $\mu : \mathrm{Spin}_n \to SO_{dim(W)}(W), \varphi \to (x \to \varphi x)$ a representation. The real spinor bundle of E is the bundle

$$S(E) := P_{Spin}(E) \times_\mu W.$$

The complex spinor bundle for $W_{\mathbb{C}}$ left module of $\mathbb{C}l_n$ and $\mu_{\mathbb{C}}$ a Spin_n representation is given by

$$S_{\mathbb{C}}(E) := P_{Spin}(E) \times_{\mu_{\mathbb{C}}} W_{\mathbb{C}},$$

Definition 3.30. We say that a spinor bundle (real or complex) is $\mathbb{Z}_2$ graded if the corresponding module W is - that means W splits into even and odd parts W^0, W^1, as Cl_n does, and those parts are respected by the multiplication. I.e., one has multiplication rules given by the Table 3.1.

$\cdot$	W^0	W^1
Cl^0	W^0	W^1
Cl^1	W^1	W^0

Table 3.1: Splitting Cl_n module structure

Remark 3.31. Since spinor bundles are bundles of modules of the Clifford bundles, the space of sections is also a module of the space of sections of the Clifford bundles.

Lastly, we discuss a $\mathbb{Z}_2$ grading of complex spinor bundles, which will be important in the following section. Let the dimension $n = 2m$ be even. From the fact of the existence of a unique irreducible representation of $\mathbb{C}l_n$ (cf. p.33 [26]), and therefore a unique irreducible module, one can conclude that there exists a unique irreducible complex spinor bundle of E. We denote it throughout this thesis by $\mathbb{S}_{\mathbb{C}}(E)$. Recall, that $\omega_{\mathbb{C}}^2 = 1$, and define $\mathbb{S}_{\mathbb{C}}(E)^{\pm}$ to be the ± 1 eigenbundles of the Clifford multiplication by $\omega_{\mathbb{C}}$. Note that for a basis vector e_i one has that $e_i\omega_{\mathbb{C}} = (-1)^{m-1}\omega_{\mathbb{C}}e_i$ and therefore using that n is even

$$e^1\omega_{\mathbb{C}} = -\omega_{\mathbb{C}}e^1, \text{ for } e^1 \in Cl^1(E) \otimes \mathbb{C}, \text{ and } e^0\omega_{\mathbb{C}} = \omega_{\mathbb{C}}e^0, \text{ for } e^0 \in Cl^0(E) \otimes \mathbb{C}.$$

Thus for $s^{\pm} \in \mathbb{S}_{\mathbb{C}}^{\pm}$:

$$e^1 s^{\pm} = e^1(\pm 1)\omega_{\mathbb{C}} s = -(\pm 1)\omega_{\mathbb{C}} e^1 s \in \mathbb{S}_{\mathbb{C}}^{\mp} \text{ and } e^0 s^{\pm} \in \mathbb{S}_{\mathbb{C}}^{\pm}.$$

That proves a splitting similar to Table 3.1 and we get a $\mathbb{Z}_2$ graded structure of $\mathbb{S}_{\mathbb{C}}$ by interpreting $\mathbb{S}_{\mathbb{C}}^{+}$ as $\mathbb{S}_{\mathbb{C}}^{0}$ and $\mathbb{S}_{\mathbb{C}}^{-}$ as $\mathbb{S}_{\mathbb{C}}^{1}$ or the other way around. We get a similar result in the real case when splitting by the real volume element. As seen in the case of the splitting of $Cl(E)$, this only works if $n \equiv 0, 3 \bmod 4$.

3.4 The Dirac Operator

3.4.1 Covariant Derivatives

For G a Lie group and a principal G-vector bundle over a manifold M, one obtains connections on the bundle (cf. A.16). Connections provide information on how to parallel transport vectors orthogonal to the fibers, i.e., horizontal vector fields. The goal of this section is to define covariant derivatives (cf. 3.32) on Clifford and spinor bundles to be able to define the Dirac operator for manifolds.
Let E be an oriented n-dimensional vector bundle over M. We start by examining connection 1-forms (cf. A.17) on $P_{SO}(E)$, which will define a covariant derivative on E. The Lie-algebra $\mathfrak{so}_n$ of SO_n consists of real, skew-symmetric $n \times n$ matrices (cf. Proposition 3.24 [14]). A 1-form on $P_{SO}(E)$ is a section of the cotangent bundle ($P_{SO}(E)$ is a manifold since M and SO_n are). Therefore, a connection 1-form ω is given by a skew-symmetric $n \times n$ matrix of 1-forms $\omega^{ij}, i, j \in \{1, ..., n\}$, i.e.

$$\begin{aligned} \omega_f : T_f P_{SO}(E) &\to \mathfrak{so}_n \\ v &\to (\omega_f^{ij}(v))_{i,j\in\{1,...,n\}}, \text{ where } \omega^{ij} = -\omega^{ji}, f \in P_{SO}(E). \end{aligned}$$

A vector $v \in T_f P_{SO}(E), f \in P_{SO}(E)$ is defined to be horizontal if and only if $\omega_f(v) = 0$. This is because $\omega_f(v)$ is an element in $\mathfrak{so}_n$, which represents the infinitesimal direction of the vector along the fiber. Therefore, the zero element in $\mathfrak{so}_n$ represents the flat connection (cf. A.17 the curvature form defined by the zero connection is zero). Covariant derivatives on E now make use of this definition of horizontality and define how vectors from a fiber of E differ from being parallel to a tangent vector of M. This is possible since we will pull back the connection 1-form ω of $P_{SO}(E)$ to M, given a local orthogonal frame. The i, j-th entry of the resulting matrix provides information about how the i-th frame vector changes with respect to the j-th frame vector along M.

Definition 3.32. A covariant derivative on E is a linear map

$$\begin{aligned} &\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \text{ such that} \\ &\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \forall f \in C^\infty(M), \sigma \in \Gamma(E). \end{aligned} \tag{3.10}$$

One obtains a map

$$\nabla_V : \Gamma(E) \to \Gamma(E) \text{ for every vector field } V \text{ on } M.$$

A covariant derivative ∇ is called Riemannian if

$$V\langle\sigma, \sigma'\rangle = \langle\nabla_V\sigma, \sigma'\rangle + \langle\sigma, \nabla_V\sigma'\rangle, \forall V \in TM, \sigma, \sigma' \in \Gamma(M), \langle\cdot,\cdot\rangle \text{ inner product on } E,$$

where the left-hand side of the equation is the derivative of the function $\langle\sigma, \sigma'\rangle$ in the direction of V. So, being Riemannian means that the covariant derivative is compatible with the inner product on E.

To define an explicit covariant derivative on E, we make use of the connection 1-form ω from above. Let $\mathcal{S} = (s_1, ..., s_n), s_i \in \Gamma(E)$ be a local family of pointwise

orthonormal sections (i.e. $s_i : U \to E, U_i \subseteq M$ open, $\langle s_i(x), s_j(x)\rangle = \delta_{ij}$ for $x \in U$). This means they also define local sections of $P_{SO}(E)$. We define a covariant derivative on $E|_U$ as

$$\nabla s_i := \sum_{j=1}^{n} \tilde{\omega}^{ji} \otimes s_j, \text{ where } \tilde{\omega} = \mathcal{S}^*\omega \tag{3.11}$$

and extended linearly with Property (3.10) for all $\sigma \in \Gamma(E)$. This local definition extends to a unique, consistent global definition of a Riemannian covariant derivative, as one can read up in Proposition 4.4 in [26]. Note that this is a natural generalization of the well-known definition of covariant derivatives on tangent bundles of Riemannian manifolds: For the Levi-Civita connection, the components of $\tilde{\omega}$ are defined via the Christoffel symbols.
Out of the existence of a Riemannian covariant derivative on E we can define a covariant derivative on the Clifford bundle $Cl(E) = P_{SO}(E) \times_{Cl(p_n)} Cl_n$ corresponding to E. Recall that $Cl(p_n) : SO_n \to Aut(Cl_n)$ is a representation, and the induced Lie algebra homomorphism is given by

$$\mathfrak{cl}(\mathfrak{p}_\mathfrak{n}) : \mathfrak{so}_\mathfrak{n} \to Der(Cl_n).$$

Let ω be a connection on $P_{SO}(E)$ as described above. We extend it to $P_{SO}(E) \times Cl_n$ by

$$\begin{aligned}\omega_{(f,\varphi)} : T_f P_{SO}(E) \oplus T_\varphi Cl_n &\to \mathfrak{so}_\mathfrak{n} \\ (v, w) &\to (\omega_f^{ij}(v))_{i,j\in\{1,\dots,n\}}, f \in P_{SO}(E), \varphi \in Cl_n\end{aligned}$$

and then push it forward to $P_{SO}(E) \times_{Cl(p_n)} Cl_n$

$$\begin{aligned}\omega'_{(f,\varphi)} : T_{[f,\varphi]}(P_{SO}(E) \times_{Cl(p_n)} Cl_n) &\to Der(Cl_n) \\ [v, w] &\to \mathfrak{cl}(\mathfrak{p}_\mathfrak{n})(\omega_f^{ij}(v))_{i,j\in\{1,\dots,n\}}), f \in P_{SO}(E), \varphi \in Cl_n.\end{aligned}$$

A covariant derivative on $Cl(E)$ can now be obtained by applying Definition (3.11) for $Cl(E)$: Let $\epsilon = (s_1, ..., s_{2^n}), s_i \in \Gamma(Cl(E))$ a local family of pointwise orthonormal sections. Recall that $dim(Cl_n) = 2^n$ and note that therefore a derivation on Cl_n can be represented as $2^n \times 2^n$ matrix. Define

$$\nabla^{Cl(E)} s_i := \sum_{j=1}^{2^n} \tilde{\omega'^{ji}} \otimes s_j, \quad \tilde{\omega'^{ji}} = \epsilon^*(\omega'^{ji})$$

and extend again linearly with property (3.10).

Proposition 3.33. *For any two sections $\sigma_i, \sigma_j \in \Gamma(Cl(E))$ it holds that*

$$\nabla^{Cl(E)}(\sigma_i \cdot \sigma_j) = (\nabla^{Cl(E)}\sigma_i) \cdot \sigma_j + \sigma_i \cdot (\nabla^{Cl(E)}\sigma_j)$$

and thus $\nabla^{Cl(E)}$ acts as a derivation on the algebra of sections of $Cl(E)$ ($\cdot$ denotes the fiberwise Clifford multiplication).

Proof. Since $\mathfrak{cl}(\mathfrak{p}_\mathfrak{n})$ maps to $Der(Cl_n)$ it holds for every $A \in \mathfrak{so}_\mathfrak{n}$ and $\varphi, \psi \in Cl_n$

$$\mathfrak{cl}(\mathfrak{p}_\mathfrak{n})(A)(\varphi\psi) = (\mathfrak{cl}(\mathfrak{p}_\mathfrak{n})(A)\varphi)\psi + \varphi(\mathfrak{cl}(\mathfrak{p}_\mathfrak{n})(A)\psi). \tag{3.12}$$

Therefore, as the pullback of ω', $\tilde{\omega'}$ also acts as a derivation on Cl_n, i.e., on the fibers of $Cl(E)$, when applying it to a tangent vector of M. This means for $x \in M, v \in T_xM$ and ϵ as above, with

$$s_i \cdot s_j = \sum_{k=1}^{2^n} f_k^{i,j} s_k, f_k^{i,j} \in C^\infty(M)$$

(to be precise, the $f_k^{i,j}$ only need to be a family of locally smooth functions, but we omit this detail for simplicity), we have

$$\tilde{\omega'}(v) \cdot (s_i s_j)(x) = (\tilde{\omega'}(v) \cdot s_i(x))s_j(x) + s_i(x)(\tilde{\omega'}(v) \cdot s_j(x))$$
$$\overset{(3.10)}{\Rightarrow} \sum_{k=1}^{2^n}(df_k^{i,j} \otimes s_k + f_k^{i,j}(\sum_{l=1}^{2^n} \tilde{\omega'^{lk}} \otimes s_l)) = (\sum_{l=1}^{2^n} \tilde{\omega'^{l,i}} \otimes s_l) \cdot s_j + s_i \cdot (\sum_{l=1}^{2^n} \tilde{\omega'^{l,j}} \otimes s_l).$$

Therefore

$$\begin{aligned}(\nabla^{Cl(E)} s_i) \cdot s_j + s_i \cdot (\nabla^{Cl(E)} s_j) &= (\sum_{l=1}^{2^n} \tilde{\omega'^{li}} \otimes s_l) \cdot s_j + s_i \cdot (\sum_{l=1}^{2^n} \tilde{\omega'^{lj}} \otimes s_l) \\ &= \sum_{k=1}^{2^n}(df_k^{i,j} \otimes s_k + f_k^{i,j}(\sum_{l=1}^{2^n} \tilde{\omega'^{lk}} \otimes s_l)) \\ &= \nabla^{Cl(E)}(s_i \cdot s_j).\end{aligned}$$

The result for all sections follows from linearity and the Leibniz rule.

□

If E carries further a spin structure $\xi : P_{Spin}(E) \to P_{SO}(E)$ consider the spinor bundle $S(E) = P_{Spin}(E) \times_\mu W$ for a module W of Cl_n. Under the covering map, a connection on $P_{SO}(E)$ lifts to a connection on $P_{Spin}(E)$. Doing the same construction as above for the Clifford bundle, we obtain

Proposition 3.34. *There is a covariant derivative $\nabla^{S(E)}$ on $S(E)$ such that it acts as a derivation with respect to the module structure:*

$$\nabla^{S(E)}(\sigma \cdot \varphi) = (\nabla^{S(E)}\sigma)\varphi + \sigma \cdot (\nabla^{S(E)}\varphi), \forall \sigma \in \Gamma(Cl(E)), \varphi \in \Gamma(S(E)).$$

Note that we obtain covariant derivatives in the complex cases by extending the derivatives in the real case $\mathbb{C}-$linearly.

3.4.2 The Dirac Operator and Dirac Bundles

From now on, let M be a Riemannian spin manifold, and let TM denote its tangent bundle. Define $Cl(M) := Cl(TM), \mathbb{C}l(M) := \mathbb{C}l(TM)$, let S be a spinor bundle over TX (real or complex), and ∇^S a corresponding (as seen in the last section) Riemannian covariant derivative.

Definition 3.35. Let $\{e_1, .., e_n\}$ be an orthonormal basis of T_xM dor $x \in M$. The Dirac operator at x is defined as the first-order differential operator

$$\begin{aligned} D_x : \Gamma(S) &\to \Gamma(S) \\ \sigma_x &\to \sum_{j=1}^{n} e_j \cdot (\nabla^S_{e_j}\sigma)_x. \end{aligned}$$

This definition should be understood as follows. Recall that $\nabla^S_{e_j}$ maps to $\Gamma(S)$. Now, since $T_xM \cong \mathbb{R}^n \hookrightarrow Cl_n$, the multiplication by e_j means (fiberwise) Clifford multiplication, and we indeed obtain an element in $\Gamma(S)$. The Dirac operator is a first-order differential operator. That means in local coordinates for any $x \in M$ (note the abuse of notation that the local coordinates of x are denoted by x), we can express D as

$$D = \sum_{j=1}^{n} A_j(x)\frac{\partial}{\partial x_j} + \text{ zero-order terms}, \tag{3.13}$$

where the $A_j(x)$ are matrices, zero-matrices are allowed. The general Expression (3.13) of a first-order differential operator should be understood as changing a section in directions with respect to the coordinates on M with coefficients determined by A_j. Let $\xi \in T^*_x(M)$ be given as $\xi = \sum_j^n \xi_j dx_j$. The principal symbol of D corresponding to ξ is defined

$$\sigma_\xi(D) := i\sum_{j=1}^{n} A_j(x)\xi_j : S_x \to S_x. \tag{3.14}$$

The Dirac operator is also elliptic, which is defined as the principal symbol $\sigma_\xi(D)$ being an isomorphism for each $\xi \neq 0$. One can see this by arguing that one can always choose local coordinates $(x_1, ..., x_n)$ on M at $x \in M$ such that x corresponds to zero, and$\{(\frac{\partial}{\partial x_1})_0, ..., (\frac{\partial}{\partial x_n})_0\}$ corresponds to the orthonormal basis $\{e_1, ..., e_n\}$ of T_xM (such coordinates are often denoted as the normal coordinates at x). An orthonormal basis of the cotangent bundle at x is then naturally given by the $dx_i|_0$. With respect to these coordinates, the connection one-form is flat at that point (the analogous statement in Riemannian geometry is that the Christoffel Symbols vanish at x if one chooses normal coordinates), since we aligned the coordinates exactly in that way. Therefore, we obtain at x and for any cotangent vector ξ at x

$$D_x = \sum_{j=1}^{n} e_j(\frac{\partial}{\partial x_j})_0 + \text{ zero-order terms}, \xi = \sum_{j=0}^{n} \xi_j(dx_j)_0 \text{ for } \xi_j \in \mathbb{R}^n.$$

Under $T_xM \cong T^*_xM$ we find the corresponding vector to ξ in the tangent space of x is given by $\sum_{j=0}^n \xi_j(\frac{\partial}{\partial x_j})_0$. This gives

$$\sigma_\xi(D_x) = i\sum_{j=1}^{n} e_j\xi_j = i\xi, \tag{3.15}$$

which denotes the map given by Clifford multiplication with $i\xi$. This defines an isomorphism, since the inverse map is given by multiplication by $\frac{-\xi}{||\xi||^2}$. Further

$$\sigma_\xi(D_x^2) = \sigma_\xi(D_x) \circ \sigma_\xi(D_x) = -\xi\xi = ||\xi||^2$$

and therefore D as well as D^2 are elliptic operators.

Example 3.36 (Continue 3.4)**.** Consider Cl_n as a module over itself and let

$$S \;=\; \mathbb{R}^n \;\times\; Cl_n$$

be the trivial module bundle over $\mathbb{R}^n$.
Case $n = 1$: We have seen in 3.4 that $Cl_1 \cong \mathbb{C}$. Choosing a basis vector e in $\mathbb{R}$ we saw that it can be interpreted as i in Cl_1. Therefore, in local coordinates, the Dirac operator on S is given by

$$D = i\frac{\partial}{\partial x_1}.$$

Case $n = 2$: $Cl_2 \cong \mathbb{H}$ and two basis vectors $e_1, e_2 \in \mathbb{R}^2$ correspond to i, j and $e_1 \otimes e_2$ to k in Cl_2. Note that decomposing $\mathbb{H}$ into $\mathbb{C} \oplus \mathbb{C}$ (as vector spaces!) corresponds to the $\mathbb{Z}_2$ grading $Cl_2 = Cl_2^0 \oplus Cl_2^1$; $\{1, e_1 \otimes e_2\}$ generate Cl_2^0 and $\{e_1, e_2\}$ generate Cl_2^1. The Dirac operator is given by

$$D = e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2}.$$

Note that e_1 and e_2 also represent matrices. Under the identification $Cl_2 \cong \mathbb{H}$ we see that D applies to functions from $\mathbb{R}^2$ to $\mathbb{C}^2 \cong \mathbb{H}$. Concretely we can identify e_1 with $\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, e_2 with $\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. One can easily check the relations $\sigma_1^2 = -I_2, \sigma_2^1 = -I_2, \sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$. Then the Dirac operator can be written as

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\frac{\partial}{\partial x_2} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix},$$

where $\frac{\partial}{\partial z} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$.

Remark 3.37. Consider a splitting of a spinor bundle $S = S^+ \oplus S^-$ (real or complex). Recall that such a splitting exists if $n \equiv 0, 3 \bmod 4$ in the real case and if n is even in the complex case. If we omit the case where $n \equiv 3 \bmod 4$, multiplication by an element e in TM swaps the subbundles: Let ω be the volume element (real or complex) and assume e is a basis vector, the general case follows by linearity. Let $v^\pm \in S^\pm$ then

$$\omega e v = (-1)^{n-1} e \omega v = \mp e v.$$

Therefore, for $n \equiv 0 \bmod 4$ in the real case and n even in the complex case, we also get a splitting of the Dirac operator D corresponding to S:

$$D^+ : \Gamma(S^+) \to \Gamma(S^-)$$
$$D^- : \Gamma(S^-) \to \Gamma(S^+)$$

Example 3.38 (Continue 3.36 *Case* $n = 2$)**.** Consider the complex spinor bundle $S_{\mathbb{C}} \cong \mathbb{R}^2 \times (Cl_2 \otimes \mathbb{C})$. Since the dimension is even, it admits a splitting corresponding to that of Cl_2, which we have seen in Example 3.36. The complex volume element is given by $\omega = ie_1e_2$, then for $v^0 \in Cl_2^0 = \{\langle 1, e_1e_2\rangle\}$ we have $v^0\omega = \omega v^0$ and for $v^1 \in Cl_2^1 = \{\langle e_1, e_2\rangle\}$ we have $v^1\omega = -\omega v^1$. Thus we get a $\mathbb{Z}_2$ grading of $S_{\mathbb{C}}$ by

$$S_{\mathbb{C}}^+ \cong \mathbb{R}^2 \times (Cl_2^0 \otimes \mathbb{C})$$
$$S_{\mathbb{C}}^- \cong \mathbb{R}^2 \times (Cl_2^1 \otimes \mathbb{C})$$

and a splitting of the corresponding Dirac operator. Note that these split Dirac operators appear in the same ways as in 3.36 but act on a different space of sections.

Next, we define Dirac bundles, which generalize spinor bundles. Recall that a spin structure is required to define spinor bundles. Via this spin structure, one can define a covariant derivative. Dirac bundles are now more generally defined on bundles of left modules of a Clifford bundle, which already inherit a covariant derivative. In particular, one can thus define a canonical associated Dirac operator on such bundles.

Definition 3.39. Let S be any (real or complex) bundle of left (real or complex) Clifford modules over M with a covariant derivative ∇^S. It is called a Dirac bundle of M if for $\langle \cdot, \cdot \rangle$ (Riemannian or Hermitian) metric on S, the following properties hold

1. $\forall x \in M, \sigma_1, \sigma_2 \in S_x, e \in T_xM$ unit vector: $\langle e\sigma_1, e\sigma_2\rangle = \langle \sigma_1, \sigma_2\rangle$ (Clifford multiplication by unit tangent vectors is orthogonal) and
2. $\forall \varphi \in \Gamma(Cl(M)), \sigma \in \Gamma(S) : \nabla^S(\varphi\sigma) = (\nabla^S\varphi)\sigma + \varphi(\nabla^S\sigma)$ (the covariant derivative acts as a derivation with respect to the module structure).

Henceforth, assume that M is closed. From the metric $\langle \cdot, \cdot \rangle$ on S one is able to define an inner product on $\Gamma(S)$ given by

$$(\sigma_1, \sigma_2) := \int_M \langle \sigma_1, \sigma_2 \rangle. \tag{3.16}$$

Via this inner product on the space of sections of a Dirac bundle, the Dirac operator is self-adjoint, i.e., $(D\sigma_1, \sigma_s) = (\sigma_1, D\sigma_2)$. This can be shown by using the definitions of the Dirac operator, a Riemannian derivative, and the properties 1 and 2 of a Dirac bundle.

Remark 3.40. Regarding the Examples 3.36 above, one might wonder if the Dirac operator generally squares to (minus) the Laplacian (also cf. 3.47). This is not true in general. Nevertheless, one obtains for E a Riemannian (Hermitian) vector bundle over M with covariant derivative ∇ and for the connection Laplacian given by

$$\nabla^*\nabla : \Gamma(E) \to \Gamma(E), \nabla^*\nabla\varphi := -\sum_{j=1}^{n} \nabla^2_{e_j,e_j}\varphi$$

where $(e_1, ..., e_n)$ orthonormal frame field of M, fulfills the following. Let D be the Dirac operator and $\nabla^*\nabla$ the connection Laplacian for any Dirac bundle S then

$$D^2 = \nabla^*\nabla + \mathcal{R},$$

with $\mathcal{R}(\varphi) := \frac{1}{2}\sum_{i,j=1}^{n} e_i e_j R_{e_j,e_k}(\varphi)$, R curvature on S (cf. Chapter 2 Theorem 8.2 [26]). Especially in the case of the Dirac operator D on a spinor bundle of M compact spin one obtains the Lichnerowicz formula (cf. Chapter 2, Theorem 8.8 [26])

$$D^2 = \nabla^*\nabla + \frac{1}{4}scal(M).$$

In particular, any compact spin manifold of positive scalar curvature has no harmonic spinors (this means $ker(D) = 0$).

Remark 3.41. The index of the Dirac operator on a Dirac bundle S $ind(D)$ is defined as the dimension of the kernel minus the dimension of the cokernel (as in general for linear maps). Since D is elliptic, these are finite numbers (to be precise, one needs to argue with the ellipticity of an extension of the Dirac operator which is bounded, cf. Chapter 3, Section 5 in [26]) and the kernel and cokernel are isomorphic: Equipping the space of sections with the inner product (3.16) yields D to be self adjoint and thus if $\sigma_1 \in coker(D) \cong im(D)^\perp$ then $0 = (\sigma_1, D\sigma_2) = (D\sigma_1, \sigma_2) \forall \sigma_2 \in \Gamma(S)$ and thus $\sigma_1 \in ker(D)$. Therefore, we can conclude that the index D must be zero, providing no information. Hence, one considers $ind(D^+)$ instead of $ind(D)$ if D splits. Note that $D^\pm$ are still elliptic and that D^+ is the adjoint of D^-, i.e. we have that $ind(D^+) = dim(ker(D^+)) - dim(ker(D^-))$.

3.4.3 Index Theorems

Let M be a closed Riemannian spin manifold of dimension $n = 4k, k \in \mathbb{N}$. Recall that the cup product is a bilinear form in $H^{2k}(M; \mathbb{Q})$ (holds in general for compact oriented manifolds). The signature sig of M is then defined to be the signature of the rational quadratic form, i.e., of $\alpha \to \alpha \smile \alpha \in H^n(M; \mathbb{Q})$ evaluated on the fundamental class $[M]$ of M, i.e., the generator of the top cohomology group. The main results in this chapter relate the (analytic) index of a Dirac operator to the so-called $\hat{A}-$ and L-genus. As obtained from the rational Pontryagin classes, the $\hat{A}-$ and $L-$ genus of a manifold are a priori homeomorphism invariants. Complete definitions will be given later in this section.

Theorem 3.42. *Consider the Dirac operator $\mathbb{D}^+ : \Gamma(\mathbb{S}^+_{\mathbb{C}}) \to \Gamma(\mathbb{S}^-_{\mathbb{C}})$ on the unique irreducible complex spinor bundle over M (cf. end of section 3.3). Then*

$$ind(\mathbb{D}^+) = \hat{A}(M).$$

Theorem 3.43. *Consider the Dirac operator $D^+_{\mathbb{C}} : \Gamma(\mathbb{C}l(M)^+) \to \Gamma(\mathbb{C}l(M)^-)$. It holds that*

$$ind(D^+_{\mathbb{C}}) = sig(M) = L(M).$$

From now on, we use the notations $\mathbb{D}, D_{\mathbb{C}}$ when referring to the operators above. Theorem 3.43 also holds more generally for M compact, oriented of even dimension,

though this is not important in our case. The Theorems directly imply that the $\hat{A}-$genus and the L-genus of a closed Riemannian spin manifold of dimension $4k$ are an integer (since clearly the index is; recall that the dimension of the kernel and cokernel of an elliptic operator are finite), giving the foundation of the definition of the Eells-Kuiper invariant in Section 5.3.1.
We start by explaining the connection between the signature of M and the index of the Dirac operator $D^+_{\mathbb{C}}$, relying on the identification $Cl(M) \cong \bigwedge^*(TM) \cong \bigwedge^*(T^*M)$ (cf. 3.27). We provide a brief overview of Hodge theory and its relations to Spin Geometry. This part follows pp. 122 in [26] and Chapter three in [20]. We can define the following for any compact, oriented Riemannian manifold of any dimension (we do not even need compactness; in that case, one needs to restrict to compactly supported forms). Still, we stay within our notation M for uniformity.
Define the volume element $dvol_x$ of $T^*_xM, x \in M$ as $\sqrt{det(g_{ij})}f_1 \wedge ... \wedge f_n$, where $f_1, .., f_n$ is a basis of T^*_xM and g_{ij} the matrix representation of the Riemannian metric g with respect to that basis. Note that the definition of $dvol_x$ is independent of the choice of an orthonormal basis since for an $n \times n$ matrix A we have, due to multilinearity, $Af_1 \wedge ... \wedge Af_n = det(A)f_1 \wedge ... \wedge f_n$ and $det(A)$ cancels out by base change of the matrix (g_{ij}).

Definition 3.44. Define a scalar product $\langle \cdot, \cdot \rangle$ on $\bigwedge^p(T^*M)$ (fiber-wise) as

$$\langle \alpha_1 \wedge ... \wedge \alpha_p, \beta_1 \wedge ... \wedge \beta_p \rangle := det(g(\alpha_i, \beta_j))_{1 \leq i \leq p, 1 \leq j \leq p}.$$

The Hodge-star operator is the linear operator

$$* : \bigwedge^p(T^*M) \to \bigwedge^{n-p}(T^*M), 0 \leq p \leq n$$

defined by

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle dvol_g, \text{ for } \alpha, \beta \in \bigwedge^p(T^*M)$$

The Hodge-star operator maps products of cotangent vectors to their orthogonal complement. One can see this by observing the following. Let $e_1, ..., e_n$ be an orthonormal basis with respect to g and $\{i_1, ..., i_n\} = \{1, ..., n\}$. Then, by the definition of the Hodge-star

$$\begin{aligned}(e_{i_1} \wedge ... \wedge e_{i_p}) \wedge (*(e_{i_1} \wedge ... \wedge e_{i_p})) &= \langle e_{i_1} \wedge ... \wedge e_{i_p}, e_{i_1} \wedge ... \wedge e_{i_p} \rangle dvol_g \\ &= det(g(e_{i_k}, e_{i_l})_{1 \leq k,l \leq p}) dvol_g \\ &= dvol_g\end{aligned}$$

and thus by comparing both sides

$$*(e_{i_1} \wedge ... \wedge e_{i_p}) = (-1)^\epsilon e_{i_{p+1}} \wedge ... \wedge e_{i_n}$$

where $\epsilon \in \{0, 1\}$ depends on how many permutations one needs from $(i_1, ..., i_n)$ to $(1, ..., n)$.
We want to examine another operator acting on the vector space of smooth forms (i.e., smooth elements in $\Gamma(\bigwedge^* T^*M)$), denoted by $\Omega^*(M)$. Recall that $\Omega^*(M)$ is graded by $\bigoplus_{p=0}^n \Omega^p(M)$. We define the L^2-norm on $\Omega^*(M)$ as

$$(\alpha, \beta) := \int_M \alpha \wedge *\beta = \int_M \langle \alpha, \beta \rangle dvol_g. \tag{3.17}$$

Definition 3.45. Denote by $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ the adjoint of the exterior derivative $d : \Omega^{p-1}(M) \to \Omega^p(M)$ with resect to (3.17) i.e.,

$$(d\alpha, \beta) = (\alpha, d^*\beta).$$

It is given by $d^* = (-1)^{n(p+1)+1} * d *$ (cf. Lemma 3.34 [20]).

Finally, we introduce the Laplace operator, acting on smooth p-forms.

Definition 3.46. The Laplace operator is given by

$$\Delta := dd^* + d^*d : \Omega^p(M) \to \Omega^p(M).$$

We call a smooth form α harmonic if $\Delta\alpha = 0$. The space of harmonic p-forms $\mathcal{H}^p$ is defined as the kernel of Δ. The space of all harmonic forms is given by $\mathcal{H} = \bigoplus_{p=0}^{n} \mathcal{H}^p$.

Recall that the exterior differential squares to zero and that there exists a cohomology theory obtained by d, called the de Rham Cohomology. It turns out that the $k-th$ de Rham Cohomology group is isomorphic to the $k-th$ singular Cohomology group and to the harmonic k-forms (as vector spaces) (cf. Chapter 3.4 in [20] and Theorem 3.14 in [28]). They therefore provide powerful information about the topology of M. One can thus think of harmonic p-forms as forms that are zero under the exterior derivative modulo the image of $(p-1)$- forms, and equivalently, as measuring holes modulo boundaries. There are now some interesting identifications of the Dirac operator D on $Cl(M)$. Also recall the formula relating D^2 to the connection Laplacian 3.40.

Lemma 3.47. *Let D be the Dirac operator corresponding to $Cl(M)$.*

1. $Cl(M) \cong \bigwedge^*(T^*M)$
2. $D \cong d + d^*$
3. $D^2 \cong \Delta$
4. $ker(D) \cong ker(\Delta) = \mathcal{H}$
5. $\omega_{\mathbb{C}} \cdot \tilde{\varphi} \cong (-1)^{k+\frac{p(p-1)}{2}} * \varphi$ *for φ $p-$form corresponding to $\tilde{\varphi}$ under 1.*

Proof. Isomorphism 1 is 3.3.
To see 2, we use the following statements. Let $x \in M$ and $(e_1, ..., e_n)$ an orthonormal frame field in a neighborhood U of M, such that $(\nabla_{e_j} e_j)_x = 0$ (take normal coordinates at x) then

$$d = \sum_{j=1}^{n} e_j \wedge \nabla_{e_j}, d^* = -\sum_{j=1}^{n} e_j \llcorner \nabla_{e_j} \tag{3.18}$$

where $\llcorner$ denotes the contraction, i.e. for $v \in T^*M, (v\llcorner) : \wedge^p(T^*M) \to \wedge^{p-1}(T^*M)$ is given by $(v\llcorner)(v_1 \wedge ... \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} \langle v_i, v \rangle v_1 \wedge ... \wedge \hat{v_i} \wedge ... \wedge v_p$. The identifications

can be seen by verifying the axioms of the exterior derivative (see Lemma 5.13 in [26]). Further under the isomorphism 1 and for $\varphi \in \bigwedge^p(T^*M)$ one has

$$\tilde{v}\tilde{\varphi} \cong v \wedge \varphi - v \llcorner \varphi, \tag{3.19}$$

where $\tilde{v}, \tilde{\varphi}$ denote the corresponding elements to v, φ in $Cl(M)$. The isomorphism (3.19) follows by showing it fiber-wise. Recall (3.27) that

$$\bigwedge^p(T^*M) \cong P_{SO}(T^*M) \times_{\wedge^p q_n} (\bigwedge^p \mathbb{R}^n)$$

and $Cl(M) = P_{SO}(T^*M) \times_{Cl(p_n)} (Cl_n)$. So let $v_x \in \mathbb{R}^n$. Choose an orthonormal basis $v_1, ..., v_n$ of $\mathbb{R}^n$ such that $v_x = tv_1, t \in \mathbb{R}$. Let $\varphi_x := v_{i_1} \wedge ... \wedge v_{i_p} \in \bigwedge^p \mathbb{R}^n$, $1 \leq i_1 < ... < i_p \leq n$. Then

$$\begin{aligned}
\tilde{v_x}\tilde{\varphi_x} &= \begin{cases} -t\tilde{v_{i_1}} \cdots \tilde{v_{i_p}}, i_1 = 1 \\ t\tilde{v_1}\tilde{v_{i_1}} \cdots \tilde{v_{i_p}}, i_1 \neq 1 \end{cases} \\
&\cong \begin{cases} -v_x \llcorner \varphi_x, i_1 = 1 \\ v_x \wedge \varphi, i_1 \neq 1 \end{cases} \\
&= v_x \wedge \varphi - v_x \llcorner \varphi_x
\end{aligned}$$

The general case for φ follows from our result on a basis. Putting together now (3.18) and (3.19) we see $D = \sum_{j=1}^n e_j \nabla_{e_j} \cong \sum_{j=1}^n e_j \wedge \nabla_{e_j} - e_j \llcorner \nabla_{e_j} = d + d^*$. Statement 3 easily follows by 2 and the fact that $d^2 = 0 = d^{*^2}$. Property 4 is just due to self-adjointness of D and thus $ker(D) = ker(D^2) \overset{3}{=} ker(\Delta)$. To see 5, we again show it on a basis, and the general case follows from that. So let $\varphi = e_1 \wedge ... \wedge e_p$ for $e_1, ..., e_p, ..., e_n$ a basis. Then $*\varphi = e_{p+1} \wedge ... \wedge e_n$ and thus

$$\begin{aligned}
\omega_{\mathbb{C}}\tilde{\varphi} &= i^{2k} e_1 \cdots e_n \cdot e_1 \cdots e_p \\
&= (-1)^k (-1)^{\frac{p(p-1)}{2}} e_{p+1} \cdots e_n \\
&= (-1)^{k + \frac{p(p-1)}{2}} * \varphi.
\end{aligned}$$

□

What follows by 4 and 5 is now that we can identify $\omega_{\mathbb{C}}$ as a map from $\mathcal{H}^p$ to $\mathcal{H}^{4k-p}, 0 \leq p \leq 2k$. Note that $\omega_{\mathbb{C}}$ is parallel, since we can choose for every x the e_j such that $(\nabla e_j)_x = 0$. So, especially, by the derivation property 3.34 $\omega_{\mathbb{C}}$ preserves the kernel of D. Thus

$$\omega_{\mathbb{C}} : \mathcal{H}^p \overset{\cong}{\to} \mathcal{H}^{4k-1} \tag{3.20}$$

is an isomorphism. Therefore, we get once more a decomposition of

$$\mathcal{H}(p) := \mathcal{H}^p \oplus \mathcal{H}^{4k-p}$$

into $\mathcal{H}^+(p) \oplus \mathcal{H}^-(p)$ for $0 \leq p < 2k$. The parts are of the same dimension, since they are constructed as the eigenspaces of $\omega_{\mathbb{C}}$ under an isomorphism. Further

$\mathcal{H}^{2k} = \mathcal{H}^{2k^+} \oplus \mathcal{H}^{2k^-}$ but these parts do not need to have the same dimension. Thus

$$\begin{aligned} ind(D^+) &= dim(ker(D^+)) - dim(ker(D^-)) \\ &= dim(\mathcal{H}^+(0)) + ... + dim(\mathcal{H}^+(2k-1)) + dim(\mathcal{H}^{2k^+}) \\ &\quad - (dim(\mathcal{H}^-(0)) + ... + dim(\mathcal{H}^-(2k-1)) + dim(\mathcal{H}^{2k^-})) \\ &= dim(\mathcal{H}^{2k^+}) - dim(\mathcal{H}^{2k^-}). \end{aligned} \tag{3.21}$$

Under $\mathcal{H}^{2k} \cong H^{2k}(M;\mathbb{R})$ and 5 in Lemma 3.47 we see that the signature of M is just (3.21):

$$\begin{aligned} sig(M) &= sig((\alpha \to \alpha \smile \alpha)[M]) \\ &\cong sig(\alpha \to \int_M \alpha \wedge \alpha) \\ &= sig(\alpha \to \begin{cases} \int_M \alpha \wedge *\alpha, \alpha \in \mathcal{H}^{2k^+} \\ \int_M \alpha \wedge - * \alpha, \alpha \in \mathcal{H}^{2k^-} \end{cases}) \\ &= sig(\alpha \to \begin{cases} \|\alpha\|^2, \alpha \in \mathcal{H}^{2k^+} \\ -\|\alpha\|^2, \alpha \in \mathcal{H}^{2k^-} \end{cases}) \\ &= dim(\mathcal{H}^{2k^+}) - dim(\mathcal{H}^{2k^-}) \end{aligned}$$

where we sloppily used the same notation α in the second line, whereas we meant the unique harmonic representatives of the α. The complex dimension of a complexified vector space is the same as the real dimension of the original vector space. Therefore, the above also proves $ind(D^+_{\mathbb{C}}) = sig(M)$ since the dimension of the kernel and cokernel of D^+, respectively $D^+_{\mathbb{C}}$, do not change when regarding them on the one hand as vector spaces over $\mathbb{R}$ and on the other hand over $\mathbb{C}$. We are left to show Theorem 3.42 and $ind(D^+_{\mathbb{C}}) = L(M)$ in 3.43. The $\hat{A}-$ and $L-$genus (in general, the definition is possible for any Riemannian manifold M) are examples of a more general concept, the total $F-$classes. Consider a formal rational power series in x with constant term 1 $f(x) \in \mathbb{Q}[[x]]^1$. Let now $x = 1 + x_1 + x_2 + ..$ be an element in any graded algebra, then we can rewrite the product $f(x_1) \cdot ... \cdot f(x_n)$ in $n \in \mathbb{Z}^+$ variables as a sum of homogeneous polynomials:

$$F(x) := (1 + F_1(x^{\underline{1}}) + F_2(x^{\underline{1}}, x^{\underline{2}}) + F_3(x^{\underline{1}}, x^{\underline{2}}, x^{\underline{3}}))(x_1, ..., x_n),$$

where $x^{\underline{k}}(x_1, ..., x_n) = \sum_{i_1 < ... < i_k} x_{i_1} \cdots x_{i_k}, 1 \leq k \leq n$ and F_k is homogeneous of degree k. We call $\{F_k(x^{\underline{1}}, ..., x^{\underline{k}})\}_{k=1}^{\infty}$ the multiplicative sequence corresponding to f.

Definition 3.48. Let E be a real vector bundle over M and $p(E) \in H^{4*}(M;\mathbb{Q})$ its total Pontryagin class (cf. C.3). Note that $H^{4*}(M;\mathbb{Q})$ is, in particular, a graded algebra. We define the total F-class corresponding to a rational power series $f(x) \in \mathbb{Q}[[x]]^1$ to be

$$F(E) := F(p(E)) \in H^{4*}(M;\mathbb{Q}),$$

where F is obtained as above.

Definition 3.49. Let $\{\hat{A}_k\}_{k=1}^{\infty}$ be the multiplicative sequence associated to

$$\hat{a}(x) = \frac{\frac{\sqrt{x}}{2}}{\sinh \frac{\sqrt{x}}{2}} = 1 - \frac{1}{24}x + \frac{7}{2^7 \cdot 3^2 \cdot 5}x^2 + ...$$

The $\hat{A}-$genus of M is then defined to be

$$\hat{A}(M) := \hat{A}(TM)[M] \in H^{4k}(M; \mathbb{Q}).$$

Remark 3.50. One defines the $\hat{A}$-genus for a manifold of dimension not a multiple of four to be zero. The first two terms of the $\hat{A}-$class are given by

$$\hat{A}_1(p_1(E)) = -\frac{1}{24}p_1(E)$$
$$\hat{A}_2(p_1(E), p_2(E)) = \frac{7}{2^7 \cdot 3^2 \cdot 5}(-4p_2(E) + 7p_1(E)^2)$$

Definition 3.51. Let $\{L_k\}_{k=1}^{\infty}, \{\hat{L}_k\}_{k=1}^{\infty}$ be the multiplicative sequences associated to

$$l(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}} = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + ...$$
$$\hat{l}(x) = l(\frac{x}{4}).$$

The L-, respectively $\hat{L}$-genus of M are defined to be

$$L(M) := L(TM)[M], \hat{L}(M) := \hat{L}(TM)[M] \in H^{4k}(M; \mathbb{Q}).$$

Remark 3.52. One defines the L- and $\hat{L}$-genus for a manifold of dimension not a multiple of four to be zero. The first two terms of the L-class are given by

$$L_1(p_1(E)) = \frac{1}{3}p_1(E)$$
$$L_2(p_1(E), p_2(E)) = \frac{1}{45}(7p_2(E) - p_1(E)^2)$$

As mentioned above, the F-classes are homomorphism invariants. But it turns out that they are also cobordism invariants.

Proposition 3.53. *The F-genus of M only depends on the oriented cobordism class of M.*

Proof. Assume that $\partial W = M$ for some compact oriented manifold W (recall M is compact, spin). The tangent space of W restricted to M is the direct sum of the tangent bundle of M with a trivial bundle, which represents the vectors pointing inside the manifold W. Since the total Pontryagin number of the trivial bundle is zero (cf. C.4), we see that $p(W)|_M = p(M)$ and thus $F(TW)|_{TM} = F(TM)$. Since M is the boundary of W, it is zero in the homology of W and therefore $F(TM)[M] = F(TW)|_{TM}[M] = 0$. □

Notably, this implies that the F-genera are additive with respect to the connected sum operation.

The Theorems 3.42 and 3.43 are now consequences of the Atiyah-Singer Index Theorem, which allows us to connect the analytical index of an operator to topology.

Theorem 3.54 (Atiyah-Singer Index Theorem)**.** *For any elliptic operator P over a closed manifold, the analytic index equals the topological index.*

For a proof and detailed explanations, we refer to [2]. The analytic index is defined as we have seen before (cf. 3.41), whereas the definition of the topological index requires some knowledge of K-theory. We follow [17] and Sections 1.9, 3.13, 3.14 in [26] based on [2]. We first consider the general case where M is a compact Hausdorff space and E is a complex vector bundle over M. The following can also be defined for real vector bundles, which are referred to as KO-theory. It turns out that K-theory gives rise to a contravariant functor from locally compact spaces to rings and can further be viewed as a cohomology theory (the cohomology theory statement is not shown here; we refer to [17]).

Definition 3.55. Let $\epsilon^m \to M$ be the trivial m-dimensional vector bundle. Two vector bundles E_1 and E_2 are stably isomorphic, denoted by $\cong_s$, if $E_1 \oplus \epsilon^m \cong E_2 \oplus \epsilon^m$ for some m. We define $K(M)$ to be the set consisting of equivalence classes of differences of vector bundles $E_1 - E_2$, where

$$E_1 - E_2 \sim E_3 - E_4 \text{ iff } E_1 \oplus E_4 \cong_s E_3 \oplus E_2.$$

Elements in $K(M)$ are constructed similarly to a ring of fractions out of the ring of isomorphism classes of vector bundles over M. Recall that a ring of fractions is constructed from a ring R by classes $[(x, y)], x, y \in R$, where

$$(x, y) \sim (u, v) \text{ iff } \exists t \in R : txv = tuy.$$

Where the t element is just needed in the case of zero divisors. $K(M)$ is indeed a ring. Addition is naturally defined as

$$(E_1 - E_2) + (E_3 - E_4) := (E_1 \oplus E_3) - (E_2 \oplus E_4)$$

and multiplication by

$$(E_1 - E_2)(E_3 - E_4) := E_1 \otimes E_3 - E_1 \otimes E_4 - E_2 \otimes E_3 + E_2 \otimes E_4.$$

The existence of inverses and transitivity of the equivalence relation $\sim$ are only guaranteed by the compactness of M (cf. Proposition 1.4 in [17]). We now obtain a contravariant functor from the category of compact Hausdorff spaces to the category of rings by

$$\begin{aligned} M &\to K(M) \\ \{f : M \to Y\} &\to \{f^* : K(Y) \to K(M), E_1 - E_2 \to f^*(E_1) - f^*(E_2)\} \end{aligned}$$

Note that f and g being homotopic implies that $f^* = g^*$. The other functorial properties also follow from the properties of pullback bundles (cf. A.11).
We now turn our attention to a setting in which base points are crucial, allowing us to define the K_{cpt} groups of locally compact Hausdorff spaces. Recall that, for example, tangent bundles of manifolds are locally compact. One can perform one-point compactifications of locally compact spaces by adding a disjoint point $\{\infty\}$ and defining an open set containing $\{\infty\}$ so that its complement is closed and compact. One usually defines ∞ as the base point of the compacification.

Definition 3.56. For M compact Hausdorff, $\tilde{K}(M)$ is defined to be the kernel of the projection $p : K(M) \to K(pt) \cong \mathbb{Z}$. As the kernel under a ring homomorphism, $\tilde{K}(M)$ is an ideal. For M locally compact Hausdorff, $K_{cpt}(M)$ is defined as $\tilde{K}(M^+)$, where M^+ is the one-point compactification of M.

By definition, we obtain an exact sequence

$$0 \to \tilde{K}(M) \to K(M) \to K(\infty) \to 0$$

and we see that $\tilde{K}(M) \oplus \mathbb{Z} \cong K(M)$ for compact spaces. As for $K(M)$, K_{cpt} is a functor, but now from locally compact Hausdorff spaces, and for base-point-preserving maps. Further an element in $K_{cpt}(M)$ can be represented by a triple $[E_1, E_2, \sigma]$. Let $[E_1 - E_2] \in K_{cpt}(M)$, then we can find a bundle map $\sigma : E_1 \to E_2$ which is an isomorphism in the preimage of a neighborhood of ∞, i.e., outside a compact set. This is due to the fact that by definition of $K_{cpt}(M)$, $[E_1 - E_2]$ lies in the kernel of $K(M^+) \to K(\infty)$ and thus $(E_1)|_\infty \cong (E_2)|_\infty$ which can be extended to an neighborhood of ∞ by the local trivializations. On the other hand, each such triple defines an element in $K_{cpt}(M)$(cf. pp.64 [26] for details). Keeping this in mind, let's go back to our initial, stronger setting of M being a closed, spin manifold of dimension $n = 4k$.
Let $\pi : T^*M \to M$ be the cotangent bundle and S be either $\mathbb{S}_{\mathbb{C}}$ or $\mathbb{C}l(M)$ with Dirac operator D. Recall that $D^+ : \Gamma(S^+) \to \Gamma(S^-)$ and the principal symbol is $\sigma_\xi(D^+) : S^+_x \to S^-_x, \xi \in T^*_x M$. Therefore, the principal symbol gives rise to a bundle map

$$\sigma^*(D^+) : \pi^*(S^+) \to \pi^*(S^-),$$

where $p^\pm : \pi^*(S^\pm) \to T^*M$ is defined as $p^\pm(s^\pm) := \pi^{-1} \circ pr^\pm(s^\pm)$, $s^\pm \in S^\pm$, $pr^\pm : S^\pm \to M$. Then, the principal symbol defines an element

$$\boldsymbol{\sigma}(D^+) := [\pi^*(S^+), \pi^*(S^-), \sigma^*(D^+)] \tag{3.22}$$

in $K_{cpt}(T^*M) \cong K_{cpt}(TM)$ (cf. Section 3.1 [26]). In general, any differential, elliptic operator defines such an element in $K_{cpt}(TM)$. The definition of the topological index of an operator now applies certain functions on this class. One of those functions is based on the so-called Thom isomorphism. Before stating the exact definition of the index, we want to introduce two versions of that isomorphism briefly. Let now $\pi : E \to M$ be a complex $m-$ dimensional vector bundle and $i : M \to E$ the zero section. Denote by H^*_{cpt} the cohomology of the complex of compactly supported chains. Since π and i are homotopy equivalences (i.e., $\pi \circ i$ and $i \circ \pi$ are

homotopic to the identity), we obtain for $0 \leq p \leq n$ induced isomorphisms

$$\pi_! := P_M^{-1} \circ \pi_* \circ P_E : H_{cpt}^{p+m}(E) \to H_{cpt}^p(M)$$
$$i_! := P_E^{-1} \circ i_* \circ P_M : H_{cpt}^p(M) \to H_{cpt}^{p+m}(E),$$

Where P_M and P_E denote the Poincaré dualities (see B.1) for M respectively E. $i_!$ is called the Thom isomorphism. There is a somewhat equivalent version of it on K_{cpt} (we will see later, cf. 3.59, what we mean by somewhat equivalent). This Thom isomorphism is given by

$$i_! : K_{cpt}(M) \to K_{cpt}(E), v \to \lambda_{-1}\pi^* v.$$

For an exact definition of λ_{-1}, further explanations and the proof that this is an isomorphism, we refer to [21] or Section 3.12 [26]. The Thom-isomorphism sends the class of a vector bundle v over M to its pullback under π multiplied by the so-called Thom-class of E, given by $\lambda_{-1} = [\pi^* \bigwedge_{\mathbb{C}}^{\text{even}} E, \pi^* \bigwedge_{\mathbb{C}}^{\text{odd}} E; \sigma]$, $\sigma_e := e \wedge -(e^*)\llcorner$, where e^* is the dual of e. Note that $\pi^* v$ is constant along the fibers. So the resulting vector bundle class in $K_{cpt}(E)$ can be thought of capturing the information of $\pi^* v$ orthogonal to the fibers times information about E along the fibers (somehow a pendant to the Euler class, see 3.59 2.).

Definition 3.57. Let $f : M \hookrightarrow \mathbb{R}^m, m > 2n$ be an embedding (cf. Whitney embedding Theorem, 6.15 in [28]). Thus $f_* : TM \hookrightarrow T\mathbb{R}^m$. Define

$$f_! : K_{cpt}(TM) \to K_{cpt}(T\mathbb{R}^m)$$

as the composition of $i_! : K_{cpt}(TM) \to K_{cpt}(N(TM))$, $N(TM)$ normal bundle of TM under f_*, and $K_{cpt}(N(TM)) \to K_{cpt}(T\mathbb{R}^m)$ by identifying $N(TM)$ with an tubular neighborhood of TM in $T\mathbb{R}^m$. Let further $q_! : K_{cpt}(T\mathbb{R}^m) \to K(pt) \cong \mathbb{Z}$. The topological index of an elliptic operator P is defined as

$$top - ind(P) := q_! f_! \boldsymbol{\sigma}(P).$$

One can show that this definition is independent of the choice of f (cf. p.244 in [26]). Since we have seen that K_{cpt} is a functor, the topological index of an operator indeed captures information about the topological behavior measured by an integer. To be precise, it measures the obstruction to extending the isomorphism at ∞ of $\sigma^*(P)$.

Our goal is now to understand how we get from the topological index of $\mathbb{D}^+$ and $D_{\mathbb{C}}^+$ to the $\hat{A}-$ and $L-$genus of M, a closed, spin manifold of dimension $n = 4k$, respectively. The Chern character ch provides such a link by defining a ring homomorphism $ch : K(M) \to H^{2*}(M; \mathbb{Q})$ (in general for M compact). Let E be a complex vector bundle over M of dimension m, and consider the total rational Chern class (cf. C)

$$c(E) = 1 + c_1(E) + ... + c_m(E).$$

Assume E splits into a direct sum of line bundles (one-dimensional vector bundles) $l_1 \oplus ... \oplus l_m$. Define x_i to be the first Chern class of the $i - th$ line bundle, then due to the Whitney product formula for Chern classes (cf. C Property 1), we get that $c(E) = \prod_{i=1}^m (1 + x_i)$ or equivalently express the $c_j(E)$ as

$x^{\underline{j}}(x_1, ..., x_m) = \sum_{1 \leq i_1 < ... < i_j \leq m} x_{i_1} \cdots x_{i_j}$. If E does not split, we can pull E back under $\mathbb{P}(E) \to M$. $\mathbb{P}(E)$ is the bundle consisting of fibers being the projective space of the fibers of E, which clearly splits as line bundles. One can show that $H^*(M;\mathbb{Z}) \to H^*(\mathbb{P}(E);\mathbb{Z})$ is injective ($H^*(\mathbb{P}(E);\mathbb{Z})$ is a free module of $H^*(M;\mathbb{Z})$, cf. C.14 in [26]), yielding to a well-defined identification of the $c_j(E)$ and $x^{\underline{j}}(x_1, ..., x_m)$.

Definition 3.58. Let x_i be such that $c(E) = 1 + x^{\underline{1}}(x_1, ..., x_m) + ... + x^{\underline{m}}(x_1, ..., x_m)$. The Chern character of E is defined as

$$ch(E) := m + ch^1(E) + ch^2(E)...$$

where $ch^i(E) := \frac{1}{i!} \sum_{j=1}^m x_j^k$, i.e.

$$ch(E) = e^{x_1} + ... + e^{x_m}.$$

Thus, it is a well-defined element in $H^{2*}(M;\mathbb{Q})$.

Just by the definition and the product formula for the Chern classes, it holds for two complex vector bundles E_1, E_2 over M that

1. $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$ and
2. $ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$.

Extending this definition to the $K-$groups by setting

$$ch([E_1 - E_2]) := ch(E_1) - ch(E_2)$$

(like extending a ring homomorphism to a homomorphism of fractions) yields a ring homomorphism

$$ch : K_{cpt}(M) \to H^{2*}(M;\mathbb{Q}).$$

This makes the Chern character applicable to $\boldsymbol{\sigma}(\mathbb{D}^+)$ and $\boldsymbol{\sigma}(D^+_{\mathbb{C}})$. We state some facts without proof (for proofs see [26] Chapter 3), which will lead to a cohomological version of the Atiyah-Singer Index Theorem.

Lemma 3.59. *Let E be a complex vector bundle over M and P an elliptic operator. Let again $i : M \to TM$ be the zero section and i^* the corresponding Thom isomorphisms. The following facts hold.*

1. *There exists a class $\mathcal{T}(E)$ such that $ch \circ i_!(\xi) = i_!(\mathcal{T}(E)ch(\xi)), \xi \in K_{cpt}(E)$.*
2. *$i^*i_!(u) = \chi(E)u, u \in H^*_{cpt}(M)$ and $i^*i_!(\xi) = \lambda_{-1}|_0(TM)\xi, \xi \in K_{cpt}(M)$.*
3. *$top - ind(P) = \{\mathcal{T}(E \otimes \mathbb{C})^{-1} ch\boldsymbol{\sigma}(P)\}[E]$*
4. *$\mathcal{T}(E \otimes \mathbb{C})^{-1} = (-1)^n \hat{A}(M)^2$*

The element $\mathcal{T}(E)$ is called difference element, since it prevents the two Thom isomorphisms $i_!$ from commuting with the Chern character. One can use 1 to show 3. Together with 4, one can prove the following cohomological version of the Atiyah-Singer Index Theorem.

Theorem 3.60 (pp.255 [26], in general for M compact,oriented)**.** *Let $\pi : TM \to M$ projection and P elliptic operator, then it holds that*

$$indP = (-1)^{\frac{n(n+1)}{2}}\{\pi_! ch\boldsymbol{\sigma}(P)\hat{A}(TM)^2\}[M].$$

We want to apply this theorem to our operators $\mathbb{D}^+$ and $D^+_{\mathbb{C}}$. So let $\pi : TM \to M$ be the bundle projection and $i : M \to TM$ the zero section. We have by (3.22) and (3.15)

$$\boldsymbol{\sigma}(D^+_{\mathbb{C}}) = [\pi^*\mathbb{C}l^+(M), \pi^*\mathbb{C}l^-(M); \psi] \in K_{cpt}(TM)$$
$$\boldsymbol{\sigma}(\mathbb{D}^+) = [\pi^*\mathbb{S}^+_{\mathbb{C}}(M), \pi^*\mathbb{S}^-_{\mathbb{C}}(M); \psi] \in K_{cpt}(TM)$$

where ψ is given by Clifford multiplication. Thus applying i^* yields elements in $K(M)$ given by

$$i^*\boldsymbol{\sigma}(D^+_{\mathbb{C}}) = [\mathbb{C}l^+(M) - \mathbb{C}l^-(M)]$$
$$i^*\boldsymbol{\sigma}(\mathbb{D}^+) = [\mathbb{S}^+_{\mathbb{C}}(M) - \mathbb{S}^-_{\mathbb{C}}(M)].$$

As above, for a complexified real vector bundle $E \otimes \mathbb{C}$ (or its lift to the bundle consisting of fibers being oriented two-dimensional subspaces of the fibers of E), one also obtains a splitting into complex line bundles, but of the form

$$E \otimes \mathbb{C} = l_1 \oplus \bar{l_1} \oplus ... \oplus l_n \oplus \bar{l_n}$$

(cf. Proposition 11.2 [26])). So to calculate $ch(i^*\boldsymbol{\sigma}(D^+_{\mathbb{C}}))$ and $ch(i^*\boldsymbol{\sigma}(\mathbb{D}^+))$, one can do so by calculating the product of the differences of the Chern characters of the line bundles. After some rewriting using the definitions of the $\hat{A}$ and $\hat{L}$ classes (cf. Proposition 11.22 and 11.24 [26]) one has

$$ch(i^*\boldsymbol{\sigma}(D^+_{\mathbb{C}})) = (-2)^n\chi(TM)\hat{L}(TM)\hat{A}(TM)^{-2} \tag{3.23}$$
$$ch(i^*\boldsymbol{\sigma}(\mathbb{D}^+)) = (-1)^n\chi(TM)\hat{A}(TM)^{-1}. \tag{3.24}$$

Now, by 3.59 2. $i^*i_!(\cdot) = \chi(TM)\cdot$ and thus

$$\chi(TM)\pi_! ch(\boldsymbol{\sigma}(D^+_{\mathbb{C}})) = i^*i_!\pi_! ch(\boldsymbol{\sigma}(D^+_{\mathbb{C}})) = i^* ch(\boldsymbol{\sigma}(D^+_{\mathbb{C}})) = ch(i^*\boldsymbol{\sigma}(D^+_{\mathbb{C}}))$$
$$\overset{(3.23)}{=} (-2)^n\chi(TM)\hat{L}(TM)\hat{A}(TM)^{-2}$$
$$\chi(TM)\pi_! ch(\boldsymbol{\sigma}(\mathbb{D}^+)) \overset{(3.24)}{=} (-1)^n\chi(TM)\hat{A}(TM)^{-1}.$$

We can now finally conclude with Theorem 3.60 and since $n = 4k$

$$ind(D^+_{\mathbb{C}}) = (-1)^{\frac{n(n+1)}{2}}(-2)^n\hat{L}(M) = L(M)$$
$$ind(\mathbb{D}^+) = (-1)^{\frac{n(n+1)}{2}}(-1)^n\hat{A}(M) = \hat{A}(M).$$

Remark 3.61. One interesting consequence of Theorem 3.42 and Remark 3.40 is, that if M admits a metric of positive scalar curvature, then $\hat{A}(M) = 0$. Since by 3.53 the $\hat{A}$-genus is an cobordism invariant we obtain a large class of manifolds not admitting positive scalar curvature which again motivates our goal to find metrics of positive sectional curvature metrics on exotic-seven spheres.

Now that we have outlined the proofs of Theorem 3.42 and 3.43, we would like to highlight similar results, but generalized for manifolds with boundary. So assume, as before, that M is an $n = 4k$ dimensional compact spin manifold, but with boundary Y. The argumentation of a well-defined index of the Dirac operator, as shortly discussed in 3.41, is based on closedness, and further, the proof of 3.54 was much simpler since the authors did not need to consider analytic boundary value problems. If one now considers boundaries, one additionally needs the assumption that M is isometric to a product near the boundary. This allows one to determine the index of the Dirac operator in terms of an integral over the interior of M, and the behavior of the operator acting on the boundary. As a result, one needs to extend the original formula by a boundary term arising from solving the global boundary problem. The resulting formula is called the Atiyah-Singer-Padori Index Theorem. For details and proofs, we refer to [1]. For the sake of this thesis, we state the explicit cases generalizing 3.42 and 3.43. Since a boundary shifts the in forhand purely topological problem to an analytic one, we interpret the Pontryagin numbers and thus the $\hat{A}-$ and $L-$ genus as (a product of) differential forms. This is achieved by identifying de Rham cocycles with cohomology classes (cf. Chern-Weil Theory, for example, Appendix C in [31], especially Corollary 1), which allows us to interpret the Pontryagin classes as closed forms in $\Omega^{4*}(M)$ and the cup product as the wedge product. Those forms are constructed from the curvature form Ω with respect to the Levi-Civita connection ∇ (cf. A.17). The forms are a priori metric-dependent; the represented characteristic classes, of course, are not. For example one has $p_1(E) = \frac{1}{8\pi^2} tr(\Omega^{E^2})$ and $ch(E) = tr(\exp \frac{-\Omega^E}{2\pi i})$, where exp is the matrix exponential and E some Hermitian vector bundle with curvature form Ω^E (see also Appendix C in [31]). In the following, we write $p_k(M)$ when referring to the k-th Pontryagin form or class of the tangent bundle $p_k(TM)$ or simply p_k if the context is clear. We also use the notations $\hat{A}$ and L simultaneously when addressing the form versions. The context should make clear how it is to be interpreted (for example, integrating makes sense only for forms, and evaluating on fundamental classes makes sense only for cohomology classes). Note, that Theorems 3.42 and 3.43 in the closed case read in the form version as $ind(\mathbb{D}^+) = \int_M \hat{A}(TM)$ and $ind(D^+_{\mathbb{C}}) = \int_M L(TM)$.

Theorem 3.62 ([1])**.** *With the assumptions above, it holds that*

$$ind(\mathbb{D}^+) = \int_M \hat{A}(TM) - \frac{dim(ker(\mathbb{D}^+|_Y)) + \eta(\mathbb{D}^+|_Y, 0)}{2} \tag{3.25}$$

$$sig(M) = \int_M L(TM) - \eta(D^+_{\mathbb{C}}|_Y, 0), \tag{3.26}$$

where η is the spectral function given by $\eta(D, z) = \sum_{\lambda \neq 0 \text{ eigenvalue of } D} \frac{sign(\lambda)}{|\lambda|^z}$, $z \in \mathbb{C}$.

Lastly, we remark on an equivariant case. So assume again M is closed, compact, spin, of dimension $n = 4k$, and additionally admits an action by a compact Lie group G. One can do the same construction as in general K-Theory, but now considering equivalence classes of G- bundles over M, i.e., complex vector bundles with a fiber-wise G action projecting to the one on M. The corresponding groups are denoted by $K_G(M)$. Similarly, one obtains a homomorphism

$$ch_G : K_G(M) \to H^*(M; \mathbb{Q}) \otimes R(G)$$

where $R(G)$ is the representation ring defined by the abelian group generated by equivalence classes of irreducible, finite-dimensional complex representations of G for elliptic, differential $G-$operators (i.e., G-equivariant operators). In the form version for E a Hermitian vector bundle over M with curvature form Ω^E, and for an orientation-preserving isometry $g : M \to M$, the equivariant Chern character reads as $ch_g(E) = tr(g \exp(\frac{-\Omega^E}{2\pi i}))$. Further it holds

$$ind_G(P) = top - ind_G(P)$$

(cf. Theorem 14.1 [26]), where P is a differential G-operator.

Chapter 4

Non-Negative Sectional Curvature on Milnor Spheres

Unless specified, this chapter is based on [13]. This chapter provides a construction of the Milnor spheres, which leads to a non-negative sectional curvature metric and can be generalized in Section 5.1. Milnor spheres are the exotic seven-spheres which can be obtained as total spaces over S^4 with fiber S^3 and structure group $SO(4)$. The goal is to show that there exists a cohomogeneity one action on every principal $SO(4)-$ bundle E over S^4 with codimension two singular orbits. Grove and Ziller showed in Chapter 2 in [13] the following result.

Theorem 4.1. *Any cohomogeneity one manifold with codimension two singular orbits admits a non-negatively curved invariant metric.*

The proof uses the decomposition of the manifold given by the Slice Theorem 2.8 and metrics on each of the two disk bundles, which match smoothly on the boundary. Together with the cohomogeneity one action on E, one obtains a non-negatively curved invariant metric on E. As a consequence, the product metric on $E \times S^3$ (S^3 with the standard round metric) now also admits a metric of non-negative curvature. Since $E \times S^3 \to E \times_{SO(4)} S^3$ is a Riemannian submersion (2.12), O'Neill's formula for submersions (2.14) then further implies a non-negative curvature metric on every S^3- bundle over S^4 with structure group $SO(4)$. Thus, every Milnor sphere admits a non-negative curvature metric. This procedure is of further importance, since we will later use a generalization of the ideas to create a metric of non-negative sectional curvature on every exotic seven-sphere.
To define a cohomogeneity one action on every principal $SO(4)-$ bundle, we define such an action on principal $S^3 \times S^3-$bundles. Notice that $S^3 \times S^3$ is the double cover of $SO(4)$. The action on $S^3 \times S^3-$bundles is defined by applying twice the construction 2.10 and explained in the following. Figure 4.4 gives an overview of the results.

We start by defining a cohomogeneity-one action of $SO(3)$ on S^4 (cf. 2.3 for the standard action of $SO(n-1)$ on S^n). Consider the five dimensional vector space $V := \{A | A \in M_3(\mathbb{R}), A = A^t, tr(A) = 0\}$ with inner product $\langle A, B\rangle = trAB$. S^4 is

a subspace of V by

$$S^4 \hookrightarrow V, (x_1, x_2, x_3, x_4, x_5) \to \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & -x_1 - x_4 \end{pmatrix}$$

and since $SO(3)$ acting via conjugation on V preserves the inner product, it also acts on S^4. Since the points in S^4 in this matrix representation are symmetric, they are orthogonal diagonalizable and thus conjugated to a matrix in $F := \{diag(\lambda_1, \lambda_2, \lambda_3) | \sum_i \lambda_i = 0, \sum_i \lambda_i^2 = 1\}$, which describes a great circle in S^4 orthogonal to the orbits. The quotient space of $SO(3)$ acting on S^4 is one-dimensional, and the orbits are determined by the eigenvalues λ_i of the matrices in S^4. To determine the orbit types of this action, we may have a look at different possibilities for the eigenvalues. If two eigenvalues are the same, the matrix must correspond to a singular orbit, since the orbit described by matrices with three different eigenvalues is larger in dimension and thus of principal type. We can for example choose $x_- = diag(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}), x_+ = diag(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$ and receive

$$\begin{aligned} K_- &= S(O(1)O(2)) \cong \{(a, A) | a \in \mathbb{Z}_2, A \in O(2), det(a)det(A) = 1\} \cong O(2), \\ K_+ &= S(O(2)O(1)) \cong O(2), \\ H &= S(O(1)O(1)O(1)). \end{aligned}$$

Lifting those groups into $S^3 = Sp(1)$, the unitary group over the quaternions, under the double cover $Sp(1) \to SO(3), q \to (x \to (qxq^{-1}))$, we obtain the following Diagram 4.1 for S^4 (cf. 2.1). Here $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the union quaternion group and

$$Pin(2) := \{e^{i\frac{\theta}{2}} | \theta \in \mathbb{R}\} \cup \{je^{i\frac{\theta}{2}} | \theta \in \mathbb{R}\}, Pjn(2) := \{e^{j\frac{\theta}{2}} | \theta \in \mathbb{R}\} \cup \{ie^{j\frac{\theta}{2}} | \theta \in \mathbb{R}\}.$$

Conjugation by $e^{i\frac{\theta}{2}} = cos(\frac{\theta}{2}) + isin(\frac{\theta}{2})$ describes a rotation by the angle θ in the $jk-$plane and conjugation by j captures the $O(1)$ component, meaning an additional reflection on the $i-$axis. This is due to $jij^{-1} = -i$. The analogous holds for the $Pjn(2)$ group after change of coordinates. Recall from Chapter 3 that all spheres are spin. Further note that under the identification $Cl_2 \cong \mathbb{H}$ from Example 3.4 that $Pin(2) \cong Pjn(2) \cong \mathrm{Pin}_2$ and $S^3 \cong Sp(1) \cong \mathrm{Spin}_3$.

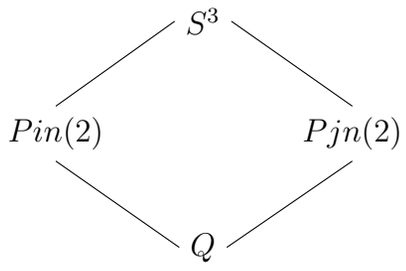


Figure 4.1: Diagram for S^4

The singular orbits are given by $S^3/Pin(2), S^3/Pjn(2)$, both of which (using the double cover) are diffeomorphic to $SO(3)/O(2) \cong \mathbb{R}\mathrm{P}^2$. Therefore, the singular orbits are of codimension two in S^4. With this cohomogeneity one action on S^4, we can construct, via Theorem 2.10, a new cohomogeneity one manifold which is a principal S^3 bundle over S^4. The assumptions of the Theorem hold trivially.

Going further, we obtain more than one cohomogeneity-one manifold from diagram 4.1. Instead of defining the homomorphism $(\phi_\pm, j_\pm) : K_\pm \to L \times G$ from Theorem 2.10, we can, because in our case $L = G$, rewrite them to $j_\pm : K_\pm \times K'_\pm \to G \times G$, where $K'_\pm \cong K_\pm$. Thus constructing new cohomogeneity one manifolds is reduced to finding copies $K'_\pm$ of $K_\pm$ such that $\Delta H \subset K_\pm \times K'_\pm$, i.e. in our case finding groups diffeomorphic to $Pin(2)$ respective $Pjn(2)$ such that the diagonal of Q is a subgroup of the product of these groups with $Pin(2)$ and $Pjn(2)$ respectively. The Diagram 4.2 describes such cohomogeneity one actions on manifolds $P_{a,b}$ by $S^3 \times S^3$ for $a, b \equiv 1 \bmod 4, a, b \neq 1$. Notice that we need $a, b \equiv 1 \bmod 4$ to ensure that $\Delta Q \subset Pjn'(2), Pin'(2)$: For example $e^{i\frac{\theta}{2}} = i \Rightarrow a = 1 \bmod 4$ in order of $e^{ia\frac{\theta}{2}} = i \Rightarrow (i, i) \in Pin'(2)$. And similar arguments hold for the other elements in ΔQ and for $Pjn'(2)$. Again, the action has codimension two singular orbits, since clearly $(S^3 \times S^3)/Pjn'(2) \cong S^3/Pjn(2), (S^3 \times S^3)/Pin'(2) \cong S^3/Pin(2)$.

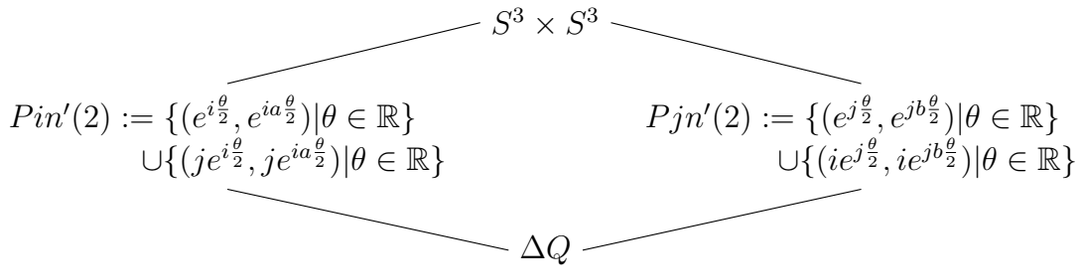


Figure 4.2: Diagram for $P_{a,b}$

Applying Construction 2.10 again, we obtain a cohomogeneity one manifold $P_{\underline{a}=(1,a_2,a_3),\underline{b}=(1,b_2,b_3)}$ which is a principal $S^3 \times S^3-$ bundle over S^4 and given by the Diagram 4.3 with the same requirements

$$a_i, b_i \equiv 1 \bmod 4, gcd(1, a_2, a_3) = 1 = gcd(1, b_2, b_3)$$

as above. With the same argument, this action has, of course, also singular orbits of codimension two.

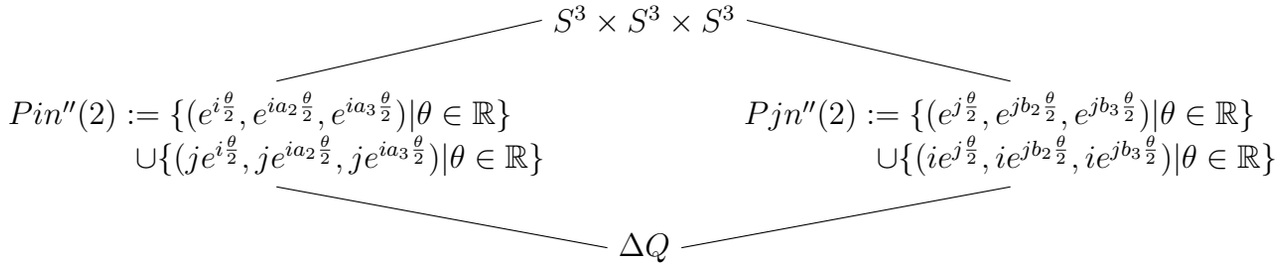


Figure 4.3: Diagram for four paramter family $P_{\underline{a},\underline{b}}$

Now, consider the double cover $S^3 \times S^3 \to SO(4), (q_1, q_2) \to (u \to q_1 u q_2^{-1})$. Under this covering we get a corresponding principal $SO(4)-$ bundle $P^*_{\underline{a},\underline{b}}$ over S^4 (the quotient of $P_{\underline{a},\underline{b}}$ by the kernel of the covering) which has a (effective) cohomogeneity one action by $SO(4) \times SO(3)$ and again codimension two singular orbits - just by construction. Grove and Ziller showed in [13] that every principal $SO(4)-$bundle

can be obtained in that way. As already mentioned, $P^*_{\underline{a},\underline{b}}$ admits a non-negatively curved metric (Theorem 4.1) and thus the associated S^3- bundle $P^*_{\underline{a},\underline{b}} \times_{SO(4)} S^3$ also does (O'Neills formula 2.14). Note that every S^3- bundle over S^4 with structure group $SO(4)$ can be obtained as an associated bundle to a principal $SO(4)$ bundle (cf. A.5).
In Chapter 5, the following observation will be of interest. One has a free action of $\{1\} \times \Delta\, S^3 \subset \{1\} \times S^3 \times S^3$ on $P_{\underline{a},\underline{b}}$ (Proof of Theorem 2.10) such that $P_{\underline{a},\underline{b}}/(\{1\} \times \Delta S^3)$ is diffeomorphic to the principal S^3-bundle

$$P^*_{\underline{a},\underline{b}} \times_{SO(4)} S^3 \overset{A.10}{\cong} P_{\underline{a},\underline{b}} \times_{S^3\times S^3} S^3 = P_{\underline{a},\underline{b}} \times_{S^3\times S^3} (S^3 \times S^3)/\Delta(S^3).$$

This is true since $P_{\underline{a},\underline{b}} \times_{S^3\times S^3} (S^3 \times S^3) \to P_{\underline{a},\underline{b}}$ is a diffeomorphism. The diffeomorphism is induced by $P_{\underline{a},\underline{b}} \times (S^3 \times S^3) \to P_{\underline{a},\underline{b}}, (p, g) \to g^{-1}p$ and $S^3 \times S^3$ acting diagonally on $P_{\underline{a},\underline{b}} \times (S^3 \times S^3)$. For the same reason as for $P^*_{\underline{a},\underline{b}}$ (Theorem 4.1), $P_{\underline{a},\underline{b}}$ does admit a non-negatively curved metric which is $S^3 \times S^3 \times S^3$ invariant and thus with 2.14 $P_{\underline{a},\underline{b}}/(\{1\} \times \Delta S^3)$ does. From this point of view, we later want to slightly manipulate the action of $\{1\} \times \Delta S^3$ to obtain a new parameter family of manifolds. This is done by allowing the first entries of $\underline{a}$ and $\underline{b}$ to be unequal one. The resulting manifolds are denoted by $M^7_{\underline{a},\underline{b}}$ (cf. 5.1). Whereas Grove and Ziller were able to classify the Milnor spheres by their appearance as principal S^3-bundles (to be precise, they classified the $P^*_{\underline{a},\underline{b}}$ by $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, cf. C.2), this is not possible for the generalized manifolds $M^7_{\underline{a},\underline{b}}$ which leads to the necessity of the Eells-Kuiper invariant in Section 5.3.1.

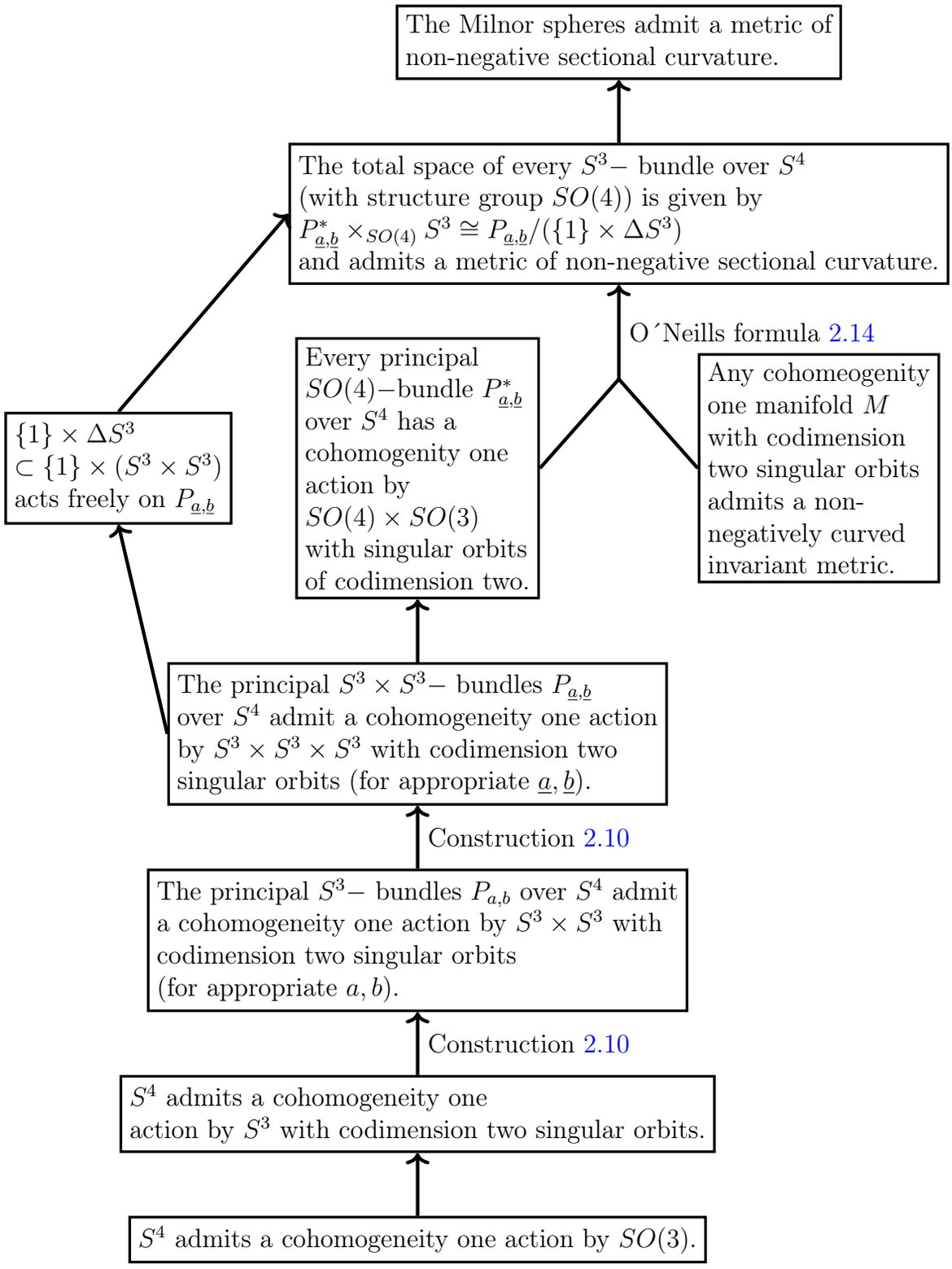


Figure 4.4: Overview non-negative curvature on Milnor spheres

Chapter 5

Non-Negative Sectional Curvature on Exotic Seven Spheres

In this chapter, all the previous results come together. As seen in Chapter 4 the Milnor spheres can be described as $P_{\underline{a},\underline{b}}/(\{1\} \times \Delta S^3)$ which leads to a metric of non-negative sectional curvature (cf. Figure 4.4). Goette, Kerin, and Shankar now extended the parameter family $P_{\underline{a},\underline{b}}$ by allowing the first entries of the triplets $\underline{a}, \underline{b}$ to be unequal to one. We will see that whenever the greatest common divisor of $a_1, a_2 \pm a_3$ and $b_1, b_2 \pm b_3$ respectively is one, then $P_{\underline{a},\underline{b}}$ has still a cohomogeneity one action and $\{1\} \times \Delta S^3$ still acts freely on $P_{\underline{a},\underline{b}}$. Thus, one obtains smooth seven-dimensional manifolds $M^7_{\underline{a},\underline{b}}$ (which are not necessarily principal S^3- bundles over S^4 anymore - but S^3- bundles over $P^{10}_{\underline{a},\underline{b}}$) that obtain a non-negative sectional curvature metric that is $SO(3)-$invariant. In Section 5.1 we lay out this construction and the proofs. In Section 5.2, we prove that the $M^7_{\underline{a},\underline{b}}$ for certain $\underline{a}, \underline{b}$ are homeomorphic to S^7. To do so, we first need to prove that for certain $\underline{a}, \underline{b}$, $M^7_{\underline{a},\underline{b}}$ has the same cohomology groups as S^7. With Whitehead's Theorem (Corollary 4.33 [15]) $M^7_{\underline{a},\underline{b}}$ is a homotopy sphere and therefore (generalized Poincaré conjecture 1.6) homeomorphic to S^7. Section 5.3 aims to distinguish the diffeomorphism types of the $M^7_{\underline{a},\underline{b}}$, especially in our case, of the homotopy spheres constructed as manifolds in this family. To do so, one uses the Eells-Kuiper invariant μ, which was introduced by Eells and Kuiper (cf. [8]). It is based on the $\hat{A}$-genus of a spin manifold (see Definition 3.49). By Theorem 3.42, this is just the index of the Dirac operator $\mathbb{D}^+$. μ somehow is a paired version of the $\hat{A}-$ genus for a spin manifold and its coboundary. Nevertheless, it is independent of the choice of the spin coboundary, as we will see. The computation of the Eells-Kuiper invariant is hard. Therefore, in Subsection 5.3.1, we focus on the definition and the proof that it actually distinguishes exotic seven-spheres. Afterwards, a formula derived from Theorem 3.62 is proven, which allows one to express the invariant in terms of Dirac operators and Pontryagin forms. Subsection 5.3.2 introduces the machinery to actually compute the Eells-Kuiper invariant for homotopy spheres. This includes defining abiabatic limits of Seifert fibrations. As homotopy spheres turn out to appear as total spaces of such fibrations, we can compute the formula from the previous subsection by calculating limits of the involved terms. In the end we obtain a simplified formula for the $\mu(M^7_{\underline{a},\underline{b}})$ only depending on $\underline{a}$ and $\underline{b}$. Therefore, one can explicitly calculate

the different $\underline{a}, \underline{b}$ for which the exotic-seven spheres are attained. If not otherwise mentioned, we follow [10] throughout this chapter.

5.1 The Manifolds $M^7_{\underline{a},\underline{b}}$

Figure 5.1 illustrates the procedure of this section.

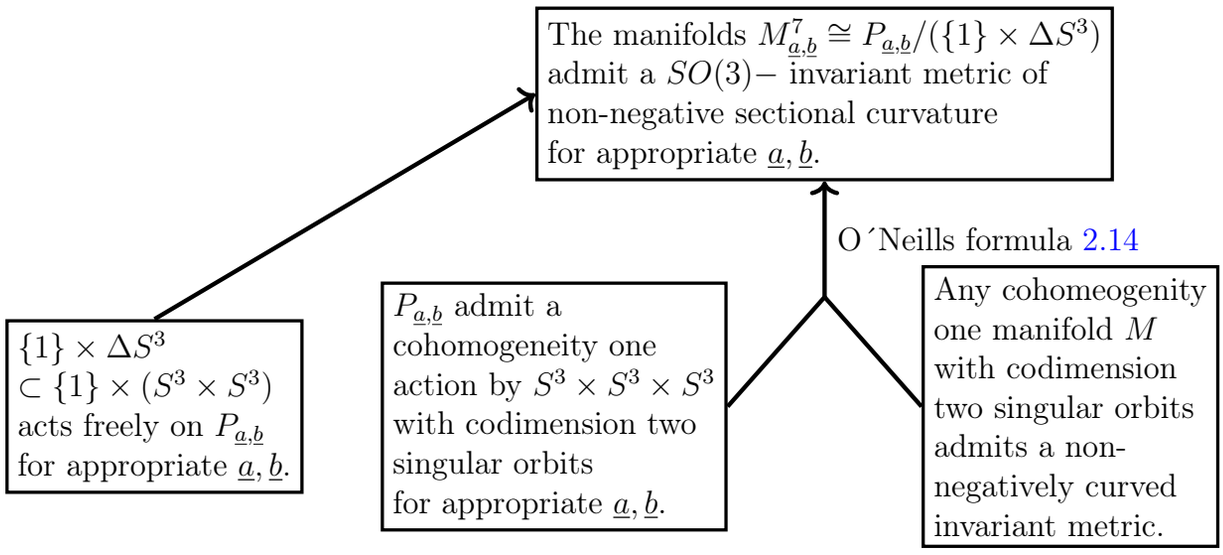


Figure 5.1: Overview non-negative curvature on $M^7_{\underline{a},\underline{b}}$

Consider the cohomogeneity one Diagram 4.3. To generalize the $P_{\underline{a},\underline{b}}$ from a four-parameter family to a six-parameter family of manifolds, we define

$$Pin(2)_{\underline{a}} := \{(e^{ia_1\frac{\theta}{2}}, e^{ia_2\frac{\theta}{2}}, e^{ia_3\frac{\theta}{2}}) | \theta \in \mathbb{R}\} \cup \{(je^{ia_1\frac{\theta}{2}}, je^{ia_2\frac{\theta}{2}}, je^{ia_3\frac{\theta}{2}}) | \theta \in \mathbb{R}\}$$
$$Pjn(2)_{\underline{b}} := \{(e^{jb_1\frac{\theta}{2}}, e^{jb_2\frac{\theta}{2}}, e^{jb_3\frac{\theta}{2}}) | \theta \in \mathbb{R}\} \cup \{(ie^{jb_1\frac{\theta}{2}}, ie^{jb_2\frac{\theta}{2}}, ie^{jb_3\frac{\theta}{2}}) | \theta \in \mathbb{R}\}$$

for $a_i, b_i \equiv 1 \bmod 4, gcd(a_1, a_2, a_3) = 1 = gcd(a_1, a_2, a_3)$ which again ensures that $Pin(2)_{\underline{a}}, Pjn(2)_{\underline{b}}$ are subgroups of $S^3 \times S^3 \times S^3$.

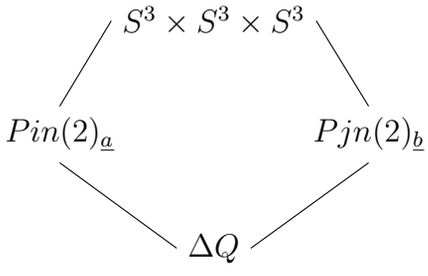


Figure 5.2: Diagram for six parameter family $P_{\underline{a},\underline{b}}$
,

It holds that $Pin(2)_{\underline{a}}/\Delta Q \cong Pjn(2)_{\underline{b}}/\Delta Q$ are diffeomorphic to S^1 and thus, as seen in Section 2.1 we obtain cohomogeneity one manifolds described by Diagram 5.2. To define the manifolds $M^7_{\underline{a},\underline{b}}$ we want to quotient the action by $\{1\} \times \Delta S^3 \subset S^3 \times S^3 \times S^3$ on $P_{\underline{a},\underline{b}}$. Recall that this is only a manifold if the action is free (a quotient of a manifold by a compact Lie group action is a manifold if the action is free, cf. Theorem 21.10 [28]).

Lemma 5.1. *$\{1\} \times \Delta S^3 \subset S^3 \times S^3 \times S^3$ acts freely on $P_{\underline{a},\underline{b}}$ if and only if $gdc(a_1, a_2 \pm a_3) = 1$ and $gdc(b_1, b_2 \pm b_3) = 1$.*

Proof. " $\Rightarrow$ "
The proof for the *gdc* conditions of $\underline{a}$ and $\underline{b}$ is analogous. We prove the first one. For a contradiction, assume that $gdc(a_1, a_2 \pm a_3) =: d \neq 1$. This implies that $e^{2\pi i \frac{a_1}{d}} = 1, e^{2\pi i \frac{a_2 - a_3}{d}} = 1$ and since $gdc(a_1, a_2, a_3) = 1$ that $q = (1, e^{2\pi i \frac{a_2}{d}}, e^{2\pi i \frac{a_2}{d}}) \in \{1\} \times \Delta S^3$ is non-trivial. But q is also an element of $Pin(2)_{\underline{a}}$:

$$1 = e^{2\pi i \frac{a_1}{d}} \Rightarrow \theta = \frac{2\pi}{d} \Rightarrow e^{2\pi i \frac{a_2}{d}} = e^{ia_2\theta}, e^{2\pi i \frac{a_2}{d}} = e^{2\pi i \frac{a_2}{d}} e^{2\pi i \frac{a_3 - a_2}{d}} = e^{2\pi i \frac{a_3}{d}} = e^{ia_3\theta}.$$

Therefore, as an element of an isotropy group, q fixes a point, and the action cannot be free.
" $\Leftarrow$ "
Assume that the *gdc* conditions hold. $\{1\} \times \Delta S^3$ acts freely on $(S^3 \times S^3 \times S^3)/\Delta Q$ since

$$\begin{aligned}(1, s, s)[q_1, q_2, q_3] = [q_1, q_2, q_3] &\Leftrightarrow (1, s, s) \in \Delta Q \\ &\Leftrightarrow s = 1, \forall s \in S^3, [q_1, q_2, q_3] \in (S^3 \times S^3 \times S^3)/\Delta Q.\end{aligned}$$

Suppose $(1, s, s) \in \{1\} \times \Delta S^3$ fixes $[q] := [q_1, q_2, q_3] \in (S^3 \times S^3 \times S^3)/Pin(2)_{\underline{a}}$. Then there exists $\alpha := (e^{ia_1\theta} = 1, e^{ia_2\theta}, e^{ia_3\theta}) \in Pin(2)_{\underline{a}}$ such that $(1, s, s)q = q\alpha$. This implies

$$q_2^{-1} q_3 = e^{-ia_2\theta} q_2^{-1} q_3 e^{ia_3\theta} \xrightarrow{q_2^{-1} q_3 = x + yj} x + yj = e^{i(a_3 - a_2)\theta} x + e^{i(a_2 + a_3)\theta} yj.$$

Together with the *gdc* condition this implies that $e^{i\theta} = 1$ and thus $q = 1$. The same holds for an element in $(S^3 \times S^3 \times S^3)/Pjn(2)_{\underline{b}}$ and therefore the action is free on every orbit, thus on the whole manifold.

□

From now on, when writing $M^7_{\underline{a},\underline{b}}$, we address the manifolds with the conditions in Lemma 5.1. As seen in the last chapter, $P_{\underline{a},\underline{b}}$ admits an $S^3 \times S^3 \times S^3$ invariant metric of non-negative curvature (Theorem 4.1). Theorem 2.12 and O'Neill's curvature formula 2.14 yield a non-negative curvature on $M^7_{\underline{a},\underline{b}} \cong P_{\underline{a},\underline{b}}/(\{1\} \times \Delta S^3)$. We get the following.

Lemma 5.2. *$M^7_{\underline{a},\underline{b}}$ admits a $SO(3)$- invariant Riemannian metric of non-negative sectional curvature.*

Proof. It remains to show the $SO(3)-$ invariance. As a subgroup of $S^3 \times S^3 \times S^3$, $S^3 \times \{(1, 1)\}$ acts isometrically on $P_{\underline{a},\underline{b}}$. Further, it clearly commutes with $\{1\} \times \Delta S^3$ and therefore the action is also isometric on $M^7_{\underline{a},\underline{b}}$. First observe that $[q_1, q_2, q_3] = [-q_1, -q_2, -q_3]$ on each orbit of $P_{\underline{a},\underline{b}}$ under the $S^3 \times S^3 \times S^3$ action, i.e., at each point of $P_{\underline{a},\underline{b}}$. Secondly we have that $[-q_1, -q_2, -q_3] = [-q_1, q_2, q_3]$ on $P_{\underline{a},\underline{b}}/(\{1\} \times \Delta S^3)$ and thus $S^3 \times \{(1, 1)\} \cong S^3$ has ineffective kernel $\{(\pm 1, 1, 1)\}$ on $M^7_{\underline{a},\underline{b}}$. Hence $M^7_{\underline{a},\underline{b}}$ admits an effective, isometric action by $SO(3) \cong S^3/\mathbb{Z}_2$. □

5.2 Cohomology of the $M^7_{\underline{a},\underline{b}}$

To compute the cohomology of $M^7_{\underline{a},\underline{b}}$, we will proceed as follows. First, we prove that $M^7_{\underline{a},\underline{b}}$ and $P^{10}_{\underline{a},\underline{b}}$ are two-connected. Together with Hurewicz and the Universal Coefficient Theorem (B.8 and B.2), this implies that the first and second cohomology groups with $\mathbb{Z}$ coefficients are trivial. Further, with Poincaré duality (Theorem B.1) and again the Universal Coefficient Theorem, we obtain

$$\begin{aligned} H^7(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong H_0(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong H^0(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong \mathbb{Z}, \\ H^6(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong H_1(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong H^1(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong 0, \\ H^5(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong H_2(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong H^2(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong 0, \\ \text{and that } H^3(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \cong Hom(H_3(M^7_{\underline{a},\underline{b}}),\mathbb{Z}) \cong Hom(H^4(M^7_{\underline{a},\underline{b}}),\mathbb{Z}) \end{aligned}$$

is the free part of $H^4(M^7_{\underline{a},\underline{b}};\mathbb{Z})$. The main work relies on proving the following Theorem.

Theorem 5.3. *Whenever* $\underline{a} = (k,-3,1), \underline{b} = (1,l,l), k,l \equiv 1 \ mod\ 4$, $H^4(M^7_{\underline{a},\underline{b}};\mathbb{Z})$ *is trivial.*

Under those conditions the cohomology groups of $M^7_{\underline{a},\underline{b}}$ are therefore isomorphic to those of S^7.

We start by proving two-connectivity.

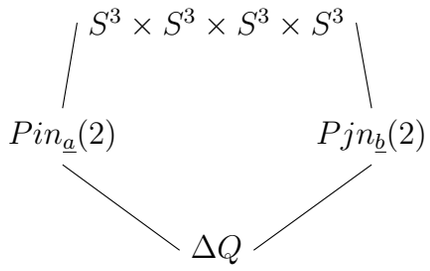


Figure 5.3: Diagram of $\hat{P}^{13}_{\underline{a},\underline{b}}$

For that we construct a new cohomogeneity one manifold $\hat{P}^{13}_{\underline{a},\underline{b}}$ out of $P^{10}_{\underline{a},\underline{b}}$ with the Procedure 2.10, $L = S^3$ and the Diagram 5.3. The tuples $\underline{a},\underline{b}$ are now four tuples $\underline{a} = (a_1,a_2,a_3,1)$, $\underline{b} = (b_1,b_2,b_3,1)$ and therefore we define $Pin_{\underline{a}}(2) := \{(e^{ia_1\frac{\theta}{2}}, e^{ia_2\frac{\theta}{2}}, e^{ia_3\frac{\theta}{2}}, e^{i\frac{\theta}{2}}) | \theta \in \mathbb{R}\}$ $\cup$ $\{(je^{ia_1\frac{\theta}{2}}, je^{ia_2\frac{\theta}{2}}, je^{ib_3\frac{\theta}{2}}, je^{i\frac{\theta}{2}}) | \theta \in \mathbb{R}\}$ and $Pjn_{\underline{b}}(2)$ analogously. Thus by 2.10 $\hat{P}^{13}_{\underline{a},\underline{b}}$ is a principal S^3- bundle over $P^{10}_{\underline{a},\underline{b}}$.

Since $S^3 \times S^3 \times S^3 \times \{1\} \subset S^3 \times S^3 \times S^3 \times S^3$ cannot be a subgroup of any of the isotropy groups (or of a conjugated isotropy group), the action by $S^3 \times S^3 \times S^3 \times \{1\}$ on $\hat{P}^{13}_{\underline{a},\underline{b}}$ is free. Recall that $P^{10}_{\underline{a},\underline{b}}$ is a principal $S^3 \times S^3$ bundle over S^4 and therefore (the actions of $\{(1,1,1)\} \times S^3$ and $S^3 \times S^3 \times S^3 \times \{1\}$ commute) $\hat{P}^{13}_{\underline{a},\underline{b}}$ is also a principal $S^3 \times S^3 \times S^3$ bundle over S^4. These two descriptions of $\hat{P}^{13}_{\underline{a},\underline{b}}$ as the total space of principal bundles prove the following Lemma. For simplicity let from now on $G := S^3 \times S^3 \times S^3$.

Lemma 5.4. $P^{10}_{\underline{a},\underline{b}}$ *and* $M^7_{\underline{a},\underline{b}}$ *are two-connected.*

Proof. We consider the long exact sequences of homotopy groups (cf. B.12) corresponding to the principal bundles discussed above.

$$\cdots \to \pi_2(S^3) \to \pi_2(\hat{P}^{13}_{\underline{a},\underline{b}}) \to \pi_2(P^{10}_{\underline{a},\underline{b}}) \to \pi_1(S^3) \to \pi_1(\hat{P}^{13}_{\underline{a},\underline{b}}) \to \pi_1(P^{10}_{\underline{a},\underline{b}}) \to 0 \tag{5.1}$$

$$\cdots \to \pi_2(G) \to \pi_2(\hat{P}^{13}_{\underline{a},\underline{b}}) \to \pi_2(S^4) \to \pi_1(G) \to \pi_1(\hat{P}^{13}_{\underline{a},\underline{b}}) \to \pi_1(S^4) \to 0 \tag{5.2}$$

Since $\pi_2(S^3) = \pi_1(S^3) = 0$ the Sequence 5.1 implies that $P^{10}_{\underline{a},\underline{b}}$ is two-connected iff $\hat{P}^{13}_{\underline{a},\underline{b}}$ is. Also $\pi_2(G) = \pi_1(G) = \pi_2(S^4) = \pi_1(S^4) = 0$ and thus Sequence 5.2 yields the two-connectness of $\hat{P}^{13}_{\underline{a},\underline{b}}$. Recall that $M^7_{\underline{a},\underline{b}}$ is a principal S^3- bundle over $P^{10}_{\underline{a},\underline{b}}$ and hence we again get a long exact sequence

$$\cdots \to \pi_2(S^3) \to \pi_2(P^{10}_{\underline{a},\underline{b}}) \to \pi_2(M^7_{\underline{a},\underline{b}}) \to \pi_1(S^3) \to \pi_1(P^{10}_{\underline{a},\underline{b}}) \to \pi_1(M^7_{\underline{a},\underline{b}}) \to 0$$

which implies that $M^7_{\underline{a},\underline{b}}$ is also two-connected. □

For the proof of Theorem 5.3, we need some discussion about the structure of $M^7_{\underline{a},\underline{b}}$. Applying the Slice Theorem 2.8 to $P^{10}_{\underline{a},\underline{b}}$ yields the decomposition (cf. 2.1)

$$P^{10}_{\underline{a},\underline{b}} = G \times_{Pin(2)_{\underline{a}}} D^2 \cup_{G/\Delta Q} G \times_{Pjn(2)_{\underline{b}}} D^2 \tag{5.3}$$

of $P^{10}_{\underline{a},\underline{b}}$ into two disk bundles over the singular orbits glued along the principal orbit. We want to achieve a similar decomposition for $M^7_{\underline{a},\underline{b}}$. For that, first note that the image of an orbit $G/U, U \in \{\Delta Q, Pin(2)_{\underline{a}}, Pjn(2)_{\underline{b}}\}$ in $M^7_{\underline{a},\underline{b}}$ is given by the biquotient $(S^3 \times S^3)//U$. The biquotient action is $(u_1, u_2, u_3)(q_1, q_2) = (q_1 u_1, u_2^{-1} q_2 u_3)$ and the following defines a diffeomorphism.

$$\begin{aligned} (G/U)/(\{1\} \times \Delta S^3) &\xrightarrow{\cong} (S^3 \times S^3)//U \\ [q_1 u_1, q_2 u_2, q_3 u_3] &\to [q_1 u_1, u_2^{-1} q_2^{-1} q_3 u_3], \quad (q_1, q_2, q_3) \in G, (u_1, u_2, u_3) \in U \end{aligned} \tag{5.4}$$

It is well defined since for $(q, q) \in \Delta S^3$ $u_2^{-1} q_2^{-1} q^{-1} q q_3 u_3 = u_2^{-1} q_2 q_3 u_3$. Injectivity follows from the definition of the biquotient action and $(q_1 u_1, q_2 u_2, q_3 u_3)$ being in the same orbit as (q_1, q_2, q_3) in $(G/U)/(\{1\} \times \Delta S^3)$. Surjectivity is clear and the map being diffeomorphic follows from S^3 being a Lie group.
Now, observe that the action of $(\{1\} \times \Delta S^3)$ is free and preserves the orbits: $(q_1 u_1, q q_2 u_2, q q_3 u_3) \in G/U$ $\forall (1, q, q) \in \{1\} \times \Delta S^3, (q_1 u_1, q_2 u_2, q_3 u_3) \in G/U$. Thus with the Decomposition (5.3) and the Diffeomorphism (5.4) one can conclude that $M^7_{\underline{a},\underline{b}}$ is the union of two disk bundles over the biquotients $(S^3 \times S^3)//Pin(2)_{\underline{a}}$ and $(S^3 \times S^3)//Pjn(2)_{\underline{b}}$ glued along the biquotient $(S^3 \times S^3)//\Delta Q$. We can argue similarly to 2.21 (consider the image of the vertical distribution on $P^{10}_{\underline{a},\underline{b}}$), to see that $M^7_{\underline{a},\underline{b}}$ is a singular Riemannian foliation (cf. 2.20) of the biquotiens. Since the action of $(\{1\} \times \Delta S^3)$ is continuous, the boundary points of the leaf space $[0, 1]$ of $P^{10}_{\underline{a},\underline{b}}$ must be send to itself. We get that the leaf space of $M^7_{\underline{a},\underline{b}}$ is also (diffeomorphic to) $[0, 1]$. $M^7_{\underline{a},\underline{b}}$ is therefor a codimension one foliation where $(S^3 \times S^3)//Pin(2)_{\underline{a}}, (S^3 \times S^3)//Pjn(2)_{\underline{b}}$ are singular leaves and $(S^3 \times S^3)//\Delta Q$ is a regular leaf. One needs to pay attention here, because since $(\{1\} \times \Delta S^3)$ does not commute with G, we do not know if the foliation originates from a global cohomogeneity one action. But what we have is that $M^7_{\underline{a},\underline{b}} = M_- \cup_{(S^3 \times S^3)//\Delta Q} M_+$ where

$$\begin{aligned} D^2 \to M_- \to (S^3 \times S^3)//Pin(2)_{\underline{a}} \\ D^2 \to M_+ \to (S^3 \times S^3)//Pjn(2)_{\underline{b}}. \end{aligned} \tag{5.5}$$

Observe that $(S^3 \times S^3)//\Delta Q$ is indeed the boundary of M_- and M_+ :

$$\begin{aligned}\partial M_- &= \partial((G \times_{Pin(2)_{\underline{a}}} D^2))/(\{1\} \times \Delta S^3) \\ &= (G/\Delta Q)/(\{1\} \times \Delta S^3) \\ &\cong (S^3 \times S^3)//\Delta Q \\ &\overset{(*)}{\cong} (S^3 \times S^3) \times_{Pin(2)_{\underline{a}}} (Pin(2)_{\underline{a}}/\Delta Q) \\ &\cong (S^3 \times S^3) \times_{Pin(2)_{\underline{a}}} S^1\end{aligned}$$

and the analogous holds for ∂M_+. To see $(*)$, consider the map

$$\begin{aligned}(S^3 \times S^3) \times_{Pin(2)_{\underline{a}}} (Pin(2)_{\underline{a}}/\Delta Q) &\to (S^3 \times S^3)//\Delta Q \\ [((s_1, s_2), pQ)]_{Pin(2)_{\underline{a}}} &\to [(s_1, s_2) \cdot p]_{\Delta Q}\end{aligned}$$

which can be checked straightforwardly to be a diffeomorphism. Therefore, we obtain two circle bundles

$$\begin{aligned}S^1 \to (S^3 \times S^3)//\Delta Q \xrightarrow{\pi_-} (S^3 \times S^3)//Pin(2)_{\underline{a}} \text{ and} \\ S^1 \to (S^3 \times S^3)//\Delta Q \xrightarrow{\pi_+} (S^3 \times S^3)//Pjn(2)_{\underline{b}}.\end{aligned}$$

The deformation retracts of the disk bundles (5.5) onto the leaves are respected by $\pi_\pm$, and we obtain with the Mayer-Vietoris-Sequence (cf. B.4)

$$\begin{aligned}\cdots \to H^2((S^3 \times S^3)//\Delta Q; \mathbb{Z}) &\to H^3(M^7_{\underline{a},\underline{b}}; \mathbb{Z}) \\ \to H^3((S^3 \times S^3)//Pin(2)_{\underline{a}}; \mathbb{Z}) &\oplus H^3((S^3 \times S^3)//Pjn(2)_{\underline{b}}; \mathbb{Z}) \\ \xrightarrow{\pi_-^* - \pi_+^*} H^3((S^3 \times S^3)//\Delta Q; \mathbb{Z}) &\to H^4(M^7_{\underline{a},\underline{b}}; \mathbb{Z}) \\ \to H^4((S^3 \times S^3)//Pin(2)_{\underline{a}}; \mathbb{Z}) &\oplus H^4((S^3 \times S^3)//Pjn(2)_{\underline{b}}; \mathbb{Z}) \to \cdots\end{aligned} \tag{5.6}$$

To calculate $H^4(M^7_{\underline{a},\underline{b}}; \mathbb{Z})$, we want to examine the cohomology groups of the singular and regular leaves. Recall that the singular leaves of $P_{\underline{a},\underline{b}}$ are $\mathbb{R}\mathrm{P}^2$ and the regular leaves are given by S^3/Q (cf. Section 4), so it might not be surprising that the following Lemmas hold.

Lemma 5.5. *Let $X = (S^3 \times S^3)//K, K \in \{Pin(2)_{\underline{a}}, Pjn(2)_{\underline{b}}\}$ be a singular leaf of $M^7_{\underline{a},\underline{b}}$. Then*

$$H^j(X; Z) = H^j(S^3 \times \mathbb{R}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0, 3, \\ 0, & j = 1, 4, \\ \mathbb{Z}_2, & j = 2, 5. \end{cases}$$

Lemma 5.6. *It holds that*

$$H^j((S^3 \times S^3)//\Delta Q; \mathbb{Z}) = H^j((S^3/Q) \times S^3; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0, 6, \\ 0, & j = 1, 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & j = 2, 5, \\ \mathbb{Z} \oplus \mathbb{Z}, & j = 3. \end{cases}$$

We want to give a short proof sketch of Lemma 5.5. Lemma 5.6 uses similar ideas. For the details, we refer to [10].

Proof sketch Lemma 5.5. Step 1: The singular orbit Y of $\hat{P}^{13}_{\underline{a},\underline{b}}$ is G-equivariant diffeomorphic to $G \times \mathbb{R}\mathrm{P}^2$.
Y is a principal G-bundle over $\mathbb{R}\mathrm{P}^2 \subset S^4$. The classifying space (cf. C.1) $BG = (\mathbb{H}\mathrm{P}^\infty)^3$ is three connected and thus a map from $\mathbb{R}\mathrm{P}^2$ into BG is null homotopic ($\mathbb{R}\mathrm{P}^2$ is two dimensional).
Step 2: Apply the Serre Spectral Sequence (B.22) to

$$Y \to X \to B(S^3 \times S^3) = \mathbb{H}\mathrm{P}^\infty \times \mathbb{H}\mathrm{P}^\infty,$$

which is the associated fibration to $S^3 \times S^3 \to Y \to X$.

$$\begin{aligned}
E_2^{0,0} &= \mathbb{Z} = E_\infty^{0,0} = H^0(X)\\
E_2^{1,0} &= H^1(\mathbb{H}\mathrm{P}^\infty \times \mathbb{H}\mathrm{P}^\infty; H^0(Y)) = 0\\
E_2^{0,1} &= H^0(\mathbb{H}\mathrm{P}^\infty \times \mathbb{H}\mathrm{P}^\infty; H^1(Y)) = 0\\
E_2^{2,0} &= H^2(B(S^3 \times S^3); H^0(Y)) = 0\\
E_2^{1,1} &= H^1(B(S^3 \times S^3); H^1(Y)) = 0\\
E_2^{0,2} &= H^0(B(S^3 \times S^3); H^2(Y)) = \mathbb{Z}_2\\
E_2^{2,1} &= H^2(B(S^3 \times S^3); H^1(Y)) = 0
\end{aligned}$$

$\Rightarrow H^0(X) = \mathbb{Z}, H^1(X) = 0, H^2(X) = \mathbb{Z}_2$.
$E_4^{0,3} = \mathbb{Z}^3, E_4^{4,0} = \mathbb{Z}^2 \Rightarrow d_4 : \mathbb{Z}^3 \to \mathbb{Z}^2 \Rightarrow ker(d_4) = E_\infty^{0,3} = H^3(X;\mathbb{Z})$
From that identification one can deduce $H^3(X;\mathbb{Z}) = \mathbb{Z}$.
Step 3 : $H^4(X;\mathbb{Z}) = 0, H^5(X;\mathbb{Z}) = \mathbb{Z}_2$.
$\mathbb{Z}^2/im(d_4) = E_\infty^{4,0} = H^4(X;\mathbb{Z})$ and the claim follows from the surjectivity of d_4.
$H^5(X;\mathbb{Z}) = \mathbb{Z}_2$ follows from $H^5(X;\mathbb{Z}) = ker(d_4 : E_4^{0,5} \to E_4^{4,2})$. □

After computing the cohomology of the leaves, we obtain from (5.6)

$$\begin{aligned}
0 \to H^3(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \to H^3((S^3 \times S^3)//Pin(2)_{\underline{a}};\mathbb{Z}) \oplus H^3((S^3 \times S^3)//Pjn(2)_{\underline{b}};\mathbb{Z})\\
\xrightarrow{\pi_-^* - \pi_+^*} H^3((S^3 \times S^3)//\Delta Q;\mathbb{Z}) \to H^4(M^7_{\underline{a},\underline{b}};\mathbb{Z}) \to 0
\end{aligned}$$

and thus

$$H^4(M^7_{\underline{a},\underline{b}};\mathbb{Z}) = coker(\pi_-^* - \pi_+^* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}).$$

The computation again uses homotopy fibrations and their Serre spectral sequences. Since the proof does not contain any insightful new ideas, we refer to [10] for the rest of the proof of Theorem 5.3.

5.3 The Diffeomorphism Types of the $M^7_{\underline{a},\underline{b}}$

5.3.1 The Eells-Kuiper Invariant

In addition to [10], this section is based on [8] and [25]. In general, the Eells-Kuiper invariant can be defined for any seven-dimensional closed, oriented, two-connected manifold with fourth cohomology being finite. We focus on the manifolds

$M^7_{\underline{a},\underline{b}}$ where $\underline{a},\underline{b}$ are as in Theorem 5.3 such that $M^7_{\underline{a},\underline{b}}$ is a homology sphere and in particular a spin manifold (the spin property can also be concluded in the general case by the universal coefficient Theorem, Poincaré duality and the vanishing second Stiefel-Whitney class). Since $\Omega_7^{Spin} = 0$ (cf. 3.25 which refers to p.92 [26]), we find an eight-dimensional spin manifold W, such that it is the coboundary of $M^7_{\underline{a},\underline{b}}$, i.e. $\partial W = M^7_{\underline{a},\underline{b}}$. Recall from the Theorems 3.42 and 3.43 that for a closed, spin manifold M of dimension $4k, k \in \mathbb{N}$ the $\hat{A}-$genus and $L-$genus, or signature, are integers. Further note that $H^m(M,\mathbb{Z}) = 0, m > 4k$ and that cohomology classes evaluated on the fundamental class of M are only non-zero if they are in the top cohomology. In dimension eight we thus obtain with 3.50, 3.52

$$sig(M) = \frac{1}{45}(-p_1(M)^2 + 7p_2(M))[M] \in \mathbb{Z} \tag{5.7}$$

$$\hat{A}(M) = \frac{1}{2^7 \cdot 45}(7p_1(M)^2 - 4p_2(M))[M] \in \mathbb{Z}. \tag{5.8}$$

Plugging in the first Pontryagin class term of (5.7) in the one of (5.8) yields

$$\hat{A}(M) = \frac{1}{2^7 \cdot 7}(p_1(M)^2[M] - 4sig(M)) \in \mathbb{Z}. \tag{5.9}$$

For a paired version of (5.9) for $M^7_{\underline{a},\underline{b}}, W$, note that by the long exact sequence of a pair we have an isomorphism

$$j : H^4(W, M^7_{\underline{a},\underline{b}}; \mathbb{Q}) \xrightarrow{\cong} H^4(W; \mathbb{Q})$$

and thus we can define the pullback of $p_1(W)$ in $H^4(W, M^7_{\underline{a},\underline{b}}; \mathbb{Q})$. Similarly we have well a well defined paired version of the signature $sig(W, M^7_{\underline{a},\underline{b}})$ (by pulling back $\alpha \to \alpha^2, \alpha \in H^4(W;\mathbb{Q})$) and a paired fundamental class $[W, M^7_{\underline{a},\underline{b}}] \in H^8(W, M^7_{\underline{a},\underline{b}}; \mathbb{Q})$.

Definition 5.7. The Elles-Kuiper invariant of $M^7_{\underline{a},\underline{b}}$ with respect to a spin coboundary W is defined as

$$\mu(M^7_{\underline{a},\underline{b}}) := \frac{1}{2^7 \cdot 7}((j^{-1}p_1(W))^2[W, M^7_{\underline{a},\underline{b}}] - 4sig(W, M^7_{\underline{a},\underline{b}})) \bmod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$$

If one compares this definition with (5.9), the intuition of its meaning is measuring the defect of being the $\hat{A}-$ genus of a closed, spin manifold, i.e., measuring the complexity of the boundary $M^7_{\underline{a},\underline{b}}$. The definition is independent of the choice of W, since the term $j^{-1}p_1(W))^2[W, M^7_{\underline{a},\underline{b}}] - 4sig(W, M^7_{\underline{a},\underline{b}})$ only differs by an integer when choosing another coboundary. This can be seen as follows. Let $(W_1, M_1), (W_2, M_2), M_1 \cong M_2 \cong M^7_{\underline{a},\underline{b}}$ be two pairs for $M^7_{\underline{a},\underline{b}}$. Let X be the closed manifold defined by the disjoint union $W_1 \sqcup_\iota -W_2$ where ι is the orientation-preserving diffeomorphism between M_1 and M_2. Consider the commutative diagram

for $r = 4, 8$.

$$\begin{array}{ccc} H^r(W_1, M^7_{\underline{a},\underline{b}}; \mathbb{Q}) \oplus H^r(W_2, M^7_{\underline{a},\underline{b}}; \mathbb{Q}) & \longleftarrow & H^r(X, M^7_{\underline{a},\underline{b}}; \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^r(W_1; \mathbb{Q}) \oplus H^r(W_2; \mathbb{Q}) & \longleftarrow & H^r(X; \mathbb{Q}) \end{array}$$

By the long exact sequence of a pair (cf B.5), the left and top maps are isomorphisms, and the right map is an epimorphism. Thus, all maps are isomorphisms, and we can conclude that

$$\begin{aligned}&j^{-1}(p_1(W_1)^2)[W_1, M^7_{\underline{a},\underline{b}}] - 4sig(W_1, M^7_{\underline{a},\underline{b}}) - (j^{-1}(p_1(W_2)^2)[W_2, M^7_{\underline{a},\underline{b}}] - 4sig(W_2, M^7_{\underline{a},\underline{b}}))\\ &= p_1(X)^2[X] - 4sig(X)\\ &= \hat{A}(X).\end{aligned}$$

From the Meyer-Vietories sequence (cf. B.4) and the fact that W_1, W_2 are spin, one can also conclude that the second Stiefel-Whitney class of X vanishes and thus X is also spin, implying its $\hat{A}$- genus to be an integer (cf. 3.42). Therefore, the Eells-Kuiper invariant of $M^7_{\underline{a},\underline{b}}$ with respect to W_1 and W_2 only differs by an integer. To see that the definition of μ is not only well-defined but indeed a tool to distinguish the diffeomorphism types of the homotopy spheres $M^7_{\underline{a},\underline{b}}$, recall that for homotopy seven-spheres being diffeomorphic is equivalent to being h-cobordant (cf. Theorem 1 [37], Definition 1.7). Further recall from Theorem 1.8 or [29] that with the connected sum operation #, the h-cobordant equivalence classes of homotopy spheres form a well-defined group Θ_7. Milnor determined in [29] that this group has exactly 28 elements. What remains to be shown is that μ is a homomorphism with respect to the connected sum and that it attains different values for different differential structures of homotopy spheres.

Theorem 5.8. *For the Eells-Kuiper invariant μ, it holds that*

1. $\mu(-M^7_{\underline{a},\underline{b}}) = -\mu(M^7_{\underline{a},\underline{b}})$
2. $\mu(M^7_{\underline{a_1},\underline{b_1}} \# M^7_{\underline{a_2},\underline{b_2}}) = \mu(M^7_{\underline{a_1},\underline{b_1}}) + \mu(M^7_{\underline{a_2},\underline{b_2}})$
3. $\mu(M^7_{\underline{a_1},\underline{b_1}}) = \mu(M^7_{\underline{a_2},\underline{b_2}})$ *if and only if* $M^7_{\underline{a_1},\underline{b_2}}, M^7_{\underline{a_2},\underline{b_2}}$ *are h-cobordant.*

Especially, for $[\cdot]$ h-cobordant equivalence class

$$\mu := \{[M^7_{\underline{a},\underline{b}}] | \underline{a} = (k, -3, 1), \underline{b} = (1, l, l), k, l \equiv 1 \ mod\ 4\} \to \mathbb{Q}/\mathbb{Z}$$

is an injective homomorphism.

Proof.

1. Follows from 2 and 3, since $M^7_{\underline{a},\underline{b}} \# - M^7_{\underline{a},\underline{b}}$ is h-cobordant to S^7 (cf. Lemma 2.4 [29]) and $\mu(S^7) = 0$ (note that D^8 is a coboundary of S^7 with trivial cohomology).

2. Let $\partial W_1 = M^7_{\underline{a_1},\underline{b_1}}$ and $\partial W_2 = M^7_{\underline{a_2},\underline{b_2}}$, then $\partial(W_1 \# W_2) = M^7_{\underline{a_1},\underline{b_1}} \# M^7_{\underline{a_2},\underline{b_2}}$ and the claim follows from the Meyer-Vietoris sequence B.4 for the connected sum (the intersection of W_1 and W_2 in $W_1 \# W_2$ is a disk) and the naturality of the Pontryagin numbers (cf. C.3): The k-th part in the sequence is

$$0 \to H^k(W_1 \# W_2; \mathbb{Q}) \xrightarrow{\cong} H^k(W_1; \mathbb{Q}) \oplus H^k(W_2; \mathbb{Q}) \to 0, 0 < k \leq n$$

and by naturality $p_k(W_1 \# W_2)|_{W_i} = p_k(W_i), i = 1, 2$.

3. Recall from Chapter 4 that for some exotic spheres, obtained as S^3 bundles over S^4, are classified by integers $(i,j) \in \mathbb{Z} \oplus \mathbb{Z} \cong \pi_3(SO(4))$ (since they appear as the associated bundles to principal $SO(4)$ bundles and apply C.2). One can show that $i + j = \pm 1$ is necessary to guarantee that the resulting bundles are homotopy spheres (cf. Chapter 4 [13]). Let M_{2i-1} be the Milnor sphere classified by (i,j). Consider the associated D^4-bundle B^8_{2i-1}, which is a spin (the second cohomology with $\mathbb{Z}_2$ coefficients is till trivial) coboundary for M_{2i-1}. Using this coboundary, computations (cf. Chapter 6 [8]) show that $\mu(M_{2i-1}) = i(i-1)/56 \bmod 1$. By 2 we see that for m copies $\mu(M^7_3 \# \cdots \# M^7_3) = m/28$. Since connected sums of manifolds which are homeomorphic to the sphere are still homeomorphic to it (cf. Lemma 2.3 [29]), we see that μ attains 28 different values for manifolds homeomorphic to the sphere. For diffeomorphic manifolds (and therefore in our case, h-cobordant manifolds), clearly, the Eells-Kuiper invariant must be the same. This is the same argument as for the independence of the boundary choice in the definition of the invariant. Therefore, μ indeed distinguishes the diffeomorphism types of the $M^7_{\underline{a},\underline{b}}$ (there are only 28).

□

The proof of 3 highly depends on the dimension, since we use the computations of Θ_7. Whereas in dimension 11 Eells and Kuiper showed an analogous result (cf. Chapter 8 [8]), in general, μ is not a diffeomorphism invariant of exotic spheres in arbitrary dimension. As a final step, we would like to highlight another expression of the Eells-Kuiper invariant that does not directly involve the coboundary but depends on the metric. This will be useful, as this metric dependency enables the use of adiabatic limits for the computations in Section 5.3.2.

Theorem 5.9. *For $(M^7_{\underline{a},\underline{b}}, g_M)$ (topology as in 5.3) and $\mathbb{D}$ the Dirac operator on the complex, irreducible spinor bundle of a coboundary W of $M^7_{\underline{a},\underline{b}}$. It holds that*

$$\mu(M^7_{\underline{a},\underline{b}}) = \frac{dim(ker(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}}})) + \eta(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}}}, 0)}{2} + \frac{1}{2^5 \cdot 7}\eta(D^+_{\mathbb{C}}|_{M^7_{\underline{a},\underline{b}}}, 0) - \frac{1}{2^7 \cdot 7}\int_{M^7_{\underline{a},\underline{b}}} p_1(M^7_{\underline{a},\underline{b}}) \wedge \hat{p_1}(M^7_{\underline{a},\underline{b}}) \ mod\ \mathbb{Z}$$

where $D^+_{\mathbb{C}}$ and η are as in 3.62, p_1 is the first Pontryagin form associated to $(M^7_{\underline{a},\underline{b}}, g_M)$ and the Levi-Civita connection as discussed in the end of Section 3.4.3 and $\hat{p_1}$ the three-form such that its derivative is p_1 on $M^7_{\underline{a},\underline{b}}$ (note that such a form exists, due to the cohomology of $M^7_{\underline{a},\underline{b}}$, it is called the primitive of p_1).

Proof. From 3.62 we get

$$sig(W) = \int_W L(TW) - \eta(D^+_{\mathbb{C}}|_{M^7_{\underline{a},\underline{b}}}, 0) \tag{5.10}$$

$$ind(\mathbb{D}^+) = \int_W \hat{A}(TW) - \frac{dim(ker(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}}})) + \eta(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}},0})}{2} \tag{5.11}$$

Without loss of generality, we can assume that $sig(W) = 0$ since we can take connected sums with the quaterionic projective space, which has signature one

and does not affect the boundary $M^7_{\underline{a},\underline{b}}$. So we can assume by (5.10) that $\int_W L(TW) = \eta(D^+_{\mathbb{C}}, 0)$. As we have computed in (5.9) we get from (5.11)

$$ind(\mathbb{D}^+) = \frac{1}{2^7 \cdot 7}\Big(\int_W p_1^2(W) - 4\eta(D^+_{\mathbb{C}}|_{M^7_{\underline{a},\underline{b}}}, 0)\Big) - \frac{dim(ker(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}}})) + \eta(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}},0})}{2}. \tag{5.12}$$

By Stokes' Theorem (cf. Theorem 16.11 [28]), one gets for the four-form p_1 and three-form $\hat{p_1}$, or their cohomological equivalent (cf. end of Section 3.4.3), that

$$\begin{aligned}
\int_W p_1^2(W) &= \int_W p_1^2(W) - \int_W d(\hat{p_1}(W) \wedge p_1(W)) + \int_{M^7_{\underline{a},\underline{b}}} \hat{p_1}(M^7_{\underline{a},\underline{b}}) \wedge p_1(M^7_{\underline{a},\underline{b}}) \\
&= \int_W (p_1(W) - d(\hat{p_1}))(W) \wedge p_1(W) + \int_{M^7_{\underline{a},\underline{b}}} \hat{p_1}(M^7_{\underline{a},\underline{b}}) \wedge p_1(M^7_{\underline{a},\underline{b}}) \\
&= ((p_1 - d(\hat{p_1}) \smile p_1)(W)[W] + \int_{M^7_{\underline{a},\underline{b}}} \hat{p_1}(M^7_{\underline{a},\underline{b}}) \wedge p_1(M^7_{\underline{a},\underline{b}}) \\
&= j^{-1}((p_1 - d(\hat{p_1})) \smile p_1)(W)[W, M^7_{\underline{a},\underline{b}}] + \int_{M^7_{\underline{a},\underline{b}}} \hat{p_1}(M^7_{\underline{a},\underline{b}}) \wedge p_1(M^7_{\underline{a},\underline{b}}) \\
&= j^{-1}(p_1^2(W))[W, M^7_{\underline{a},\underline{b}}] + \int_{M^7_{\underline{a},\underline{b}}} \hat{p_1}(M^7_{\underline{a},\underline{b}}) \wedge p_1(M^7_{\underline{a},\underline{b}}),
\end{aligned}$$

where the second equation holds since $d(p_1) = d^2(\hat{p_1}) = 0$. The second-to-last equation is valid due to p_1 and $d(\hat{p_1})$ coinciding on $M^7_{\underline{a},\underline{b}}$. Thus, we obtain by plugging into (5.12) :

$$\begin{aligned}
\mu(M^7_{\underline{a},\underline{b}}) &= \frac{1}{2^7 \cdot 7}((j^{-1}p_1(W))^2[W, M^7_{\underline{a},\underline{b}}] - 4sig(W, M^7_{\underline{a},\underline{b}})) \bmod \mathbb{Z} \\
&= ind(\mathbb{D}^+) + \frac{dim(ker(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}}})) + \eta(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}},0})}{2} \\
&+ \frac{1}{2^7 \cdot 7}(4\eta(D^+_{\mathbb{C}}|_{M^7_{\underline{a},\underline{b}}}, 0) - \int_{M^7_{\underline{a},\underline{b}}} \hat{p_1}(M^7_{\underline{a},\underline{b}}) \wedge p_1(M^7_{\underline{a},\underline{b}}) - 4sig(W, M^7_{\underline{a},\underline{b}})) \bmod \mathbb{Z} \\
&= \frac{dim(ker(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}}})) + \eta(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}},0})}{2} + \frac{1}{2^5 \cdot 7}\eta(D^+_{\mathbb{C}}|_{M^7_{\underline{a},\underline{b}}}, 0) \\
&- \frac{1}{2^7 \cdot 7}\int_{M^7_{\underline{a},\underline{b}}} p_1(M^7_{\underline{a},\underline{b}}) \wedge \hat{p_1}(M^7_{\underline{a},\underline{b}}) \bmod \mathbb{Z}
\end{aligned}$$

where for the last equation we have used that $sig(W) = 0$ and j is an isomorphism, thus $sig(W, M^7_{\underline{a},\underline{b}}) = 0$. □

5.3.2 The Computation of the Eells-Kuiper Invariant for the Manifolds $M^7_{\underline{a},\underline{b}}$

In addition to [10], the following is also based on [9].

The main goal of this section is to apply the following formula for the homotopy spheres $M^7_{\underline{a},\underline{b}}$.

Corollary 5.10. *Let $\pi : (M^7_{\underline{a},\underline{b}}, g_{M^7_{\underline{a},\underline{b}}}) \to (B^4_{\underline{a},\underline{b}}, g_{B^4_{\underline{a},\underline{b}}})$ be a Riemannian submersion onto a four-dimensional orbifold such that the fibers are S^3, totally geodesic and have positive scalar curvature. Then for $\pi^*\hat{p_1}(B^4_{\underline{a},\underline{b}})$ the three form with $d\pi^*\hat{p_1}(B^4_{\underline{a},\underline{b}}) = \pi^*p_1(B^4_{\underline{a},\underline{b}}) = p_1(\pi^*(B^4_{\underline{a},\underline{b}}))$ (which exists due to vanishing fourth cohomology)*

$$\begin{aligned}\mu(M^7_{\underline{a},\underline{b}}) = &\frac{1}{2}\int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{A}_{\Lambda B^4_{\underline{a},\underline{b}}}(TB^4_{\underline{a},\underline{b}})2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(\mathbb{D}^+|_{S^3}) \\ &+ \frac{1}{2^5\cdot 7}\int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{L}_{\Lambda B^4_{\underline{a},\underline{b}}}(TB^4_{\underline{a},\underline{b}})2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(D^+_{\mathbb{C}}|_{S^3}) + \frac{1}{2^5\cdot 7}\lim_{\epsilon\to 0}\tau_\epsilon \\ &- \frac{1}{2^7\cdot 7}\int_{M^7_{\underline{a},\underline{b}}} (p_1(\mathcal{V}) + \pi^*p_1(B^4_{\underline{a},\underline{b}})) \wedge (\hat{p}_1(\mathcal{V}) + \pi^*\hat{p}_1(B^4_{\underline{a},\underline{b}})) \quad mod\ \mathbb{Z}\end{aligned}$$

where $\Lambda B^4_{\underline{a},\underline{b}}$ is the inertia orbifold of $B^4_{\underline{a},\underline{b}}$ (cf. 2.28), $\hat{A}_{\Lambda B^4_{\underline{a},\underline{b}}}, \hat{L}_{\Lambda B^4_{\underline{a},\underline{b}}}$ as in 5.14 and $\eta_{\Lambda B^4_{\underline{a},\underline{b}}}, \tau_\epsilon$ as explained in 5.16. All forms are with respect to the corresponding Levi-Civita connection, $\mathcal{V}$ denotes the vertical subbundle corresponding to π, and $\mathbb{D}^+|_{S^3}, D^+_{\mathbb{C}}|_{S^3}$ are the Dirac operators as in Theorem 5.9 restricted to the fiber.

The proof involves so-called adiabatic limits of metrics. We introduce the relevant definitions, the choice of an orbifold $B^4_{\underline{a},\underline{b}}$, and suitable metrics. Note that the metrics on the $M^7_{\underline{a},\underline{b}}$ do not have to coincide with those in 5.2 due to the initial metric independence of the definition of μ. Thus, a suitable metric for the computation can be chosen. Corollary 5.10 then implies the final expression:

Corollary 5.11.

$$\mu(M^7_{\underline{a},\underline{b}}) = \frac{1 - a_1^2b_1^2m^2}{2^5\cdot 7} - D(\underline{a}) + D(\underline{b}) \ mod\ \mathbb{Z}$$

where $D(\underline{a}) = D(a_1; 4, a_2+a_3, a_2-a_3), D(\underline{b}) = D(b_1; 4, b_2+b_3, b_2-b_3)$ are generalized Dedekind sums, i.e., given by

$$D(x; y_1, y_2, y_3) = \frac{1}{2^6\cdot 7\cdot x^2}\sum_{l=0}^{|q|-1}\sum_{\substack{(i,j,k)\ cyclic \\ permutation\ of\ (1,2,3)}} p_i\left(\frac{14cos(\frac{p_i\pi l}{q}) + cos(\frac{p_j\pi l}{q})cos(\frac{p_k\pi l}{q})}{sin^2(\frac{p_i\pi l}{q})sin(\frac{p_j\pi l}{q})sin(\frac{p_k\pi l}{q})}\right)$$

and

$$m = \frac{1}{8a_1^2b_1^2}det\begin{pmatrix} a_1^2 & b_1^2 \\ a_2^2+a_3^2+8 & b_2^2+b_3^2+8\end{pmatrix}$$

We aim to explain the terms in the formulas and the used strategies, but provide no detailed proofs; for that, we refer to Chapter 3 in [10]. The Diagram 5.4 gives an overview of the proof strategy.

We start by introducing adiabatic limits. These are certain limits of metrics, and can thus also be applied to invariants that depend on a metric. Assume that the conditions of Corollary 5.10 hold. The idea is to blow up the base space, or

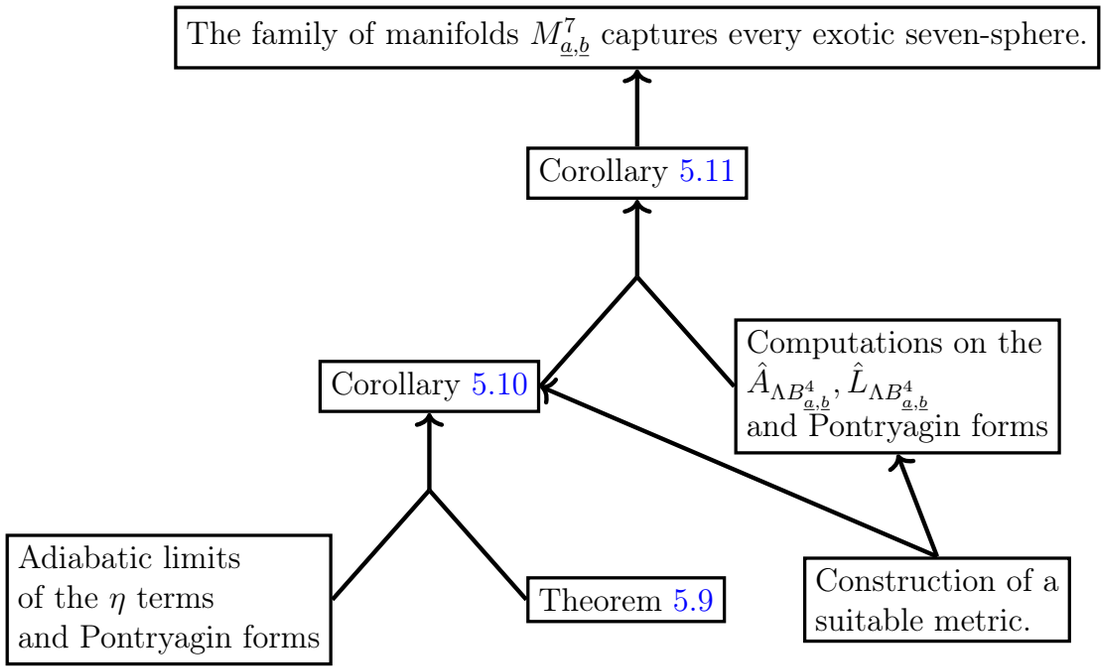


Figure 5.4: Overview on calculating μ

equivalently, shrink the fibers by some factor $\epsilon^2, \epsilon > 0$. To be precise we want to obtain a family of metrics $g_{M^7_{\underline{a},\underline{b}},\epsilon}$ on $M^7_{\underline{a},\underline{b}}$ such that

$$g_{M^7_{\underline{a},\underline{b}},\epsilon|_{\mathcal{V}}} = g_{M^7_{\underline{a},\underline{b}}|_{\mathcal{V}}}, g_{M^7_{\underline{a},\underline{b}},\epsilon|_{\mathcal{H}}} = \epsilon^{-2} g_{M^7_{\underline{a},\underline{b}}|_{\mathcal{H}}},$$

for $\mathcal{V}$ vertical and $\mathcal{H}$ horizontal subspace. Then $\lim_{\epsilon\to 0}(M^7_{\underline{a},\underline{b}}, g_{M^7_{\underline{a},\underline{b}},\epsilon})$ is called the adiabatic limit. It can be thought of as shrinking the fibers infinitely small. In general, for any Riemannian submersion $\pi : (M, g_M) \to (B, g_B)$ fulfilling the assumptions of Corollary 5.10 one similarly defines the adiabatic limit of a family of metrics $(g_{M,\epsilon})_{\epsilon\geq 0}$. Consider an orthonormal frame $(e_1, ..., e_4)$ of TB and their horizontal lifts $(\tilde{e}_1, ..., \tilde{e}_4)$ under π, i.e. an orthonormal frame of the horizontal sub-bundle. Then, for (f_1, f_2, f_3) a local orthonormal frame of the vertical sub-bundle, an orthonormal frame for $g_{M,\epsilon}$ can be defined as

$$e_{1,\epsilon} = f_1, e_{2,\epsilon} = f_2, e_{3,\epsilon} = f_3, e_{4,\epsilon} = \epsilon\tilde{e}_1, ..., e_{7,\epsilon} = \epsilon\tilde{e}_4. \tag{5.13}$$

Indeed $g_{M,\epsilon}(e_{i,\epsilon}, e_{j,\epsilon}) = \epsilon^2 g_{M,\epsilon}(\tilde{e}_{i-3}, \tilde{e}_{j-3}) = g_M(\tilde{e}_{i-3}, \tilde{e}_{j-3}) = \delta_{i,j}, i, j \in 4, ..., 7$ and orthonormality between the other sections is clear. Next, we define a family of Dirac operators corresponding to $g_{M,\epsilon}$ such that the fiberwise kernels form an orbi-bundle.

Definition 5.12. Let $\pi : (M, g_M) \to (B, g_B)$ fulfill the assumptions in 5.10 and $(g_{M,\epsilon})_{\epsilon\geq 0}$ be an adiabatic family of metrics with orthonormal frame (5.13). An adiabatic family of Dirac bundles of π is an Hermitian vector bundle (E, g_E) over (M, g_M) together with a Clifford multiplication $c : TM \to End(E)$ and a family of connections $(\nabla^{E,\epsilon})_{\epsilon\geq 0}$ such that

1. $\forall\epsilon > 0, (E, \nabla^{E,\epsilon}, g_E, c_\epsilon)$ is a Dirac bundle on $(M, g_{M,\epsilon})$ with Dirac operator $D_{M,\epsilon}$, where $c_\epsilon(e_{i,\epsilon}) := c(e_{i,1}), 1 \leq i \leq 7$.

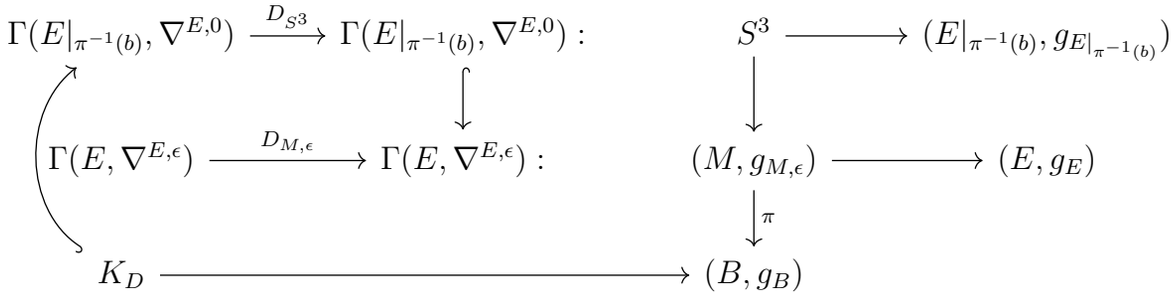


Figure 5.5: Illustration adiabatic family of Dirac operators and bundles.

2. $\nabla^{E,\epsilon}$ is analytic in ϵ around $\epsilon = 0$.

3. Let K_D be the kernels of the fiberwise Dirac operators

$$D_{S^3} = \sum_{i=1}^{3} c(f_i)\nabla^{E,0}_{f_i}$$

acting on sections of $E|_{\pi^{-1}(b)}, b \in B$, then $K_D \to B$ is a vector orbi-bundle.

We call $(D_{M,\epsilon})_{\epsilon\geq 0} = (\sum_{i=1}^{7} c_\epsilon(e_{i,\epsilon})\nabla^{E,\epsilon}_{e_{i,\epsilon}})_{\epsilon\geq 0}$ an adiabatic family of Dirac operators corresponding to π. See Diagram 5.5 for an illustration

Property 2 ensures that the Dirac operators $D_{M,\epsilon}$ are analytic and thus that there exists some $\epsilon_0 > 0$ such that the dimension of their kernels is constant for $0 < \epsilon < \epsilon_0$. One can show that this implies that there are finitely many very small eigenvalues of $D_{M,\epsilon}, 0 < \epsilon < \epsilon_0$ (see 2.g in [9]). That means there are finitely many eigenvalues unequal to zero and in $\mathcal{O}(\epsilon^2)$.

Since we can choose a metric for $M^7_{\underline{a},\underline{b}}$ in Theorem 5.9, we can prove Corollary 5.10 by considering adiabatic limits. The adiabatic limits of the η functions of the adiabatic family of Dirac operators exist due to Property 3 in 5.12 (cf. [9]) and are given by globally extended versions of the local equivariant $\hat{A}$- (and $\hat{L}$-) forms to the whole inertia orbifold.

Definition 5.13. Let M be a Riemannian spin manifold of dimension $4k$. Denote by M^g the fixed point set of an isometry $g \in G$, G a compact Lie group, on M given by multiplication. Consider the normal bundle $N_g \to M^g$ to M^g of rank $2k_g$ (M us oriented). g has an associated representation in $SO(2k_g)$ acting on the fibers of N_g, since it acts trivially on M^g but effectively on M, which lifts to an action given by $\tilde{g} \in Spin(2k_g)$. Consider the irreducible spinor bundle $\mathbb{S}_{\mathbb{C}}(N_g) = \mathbb{S}^+_{\mathbb{C}}(N_g) \oplus \mathbb{S}^-_{\mathbb{C}}(N_g)$ of N_g. The equivariant $\hat{A}-$form to an isometric action g, is defined as

$$\hat{A}_{\tilde{g}}(TM) = (-1)^{k_g}\frac{\hat{A}(TM^g)}{ch_{\tilde{g}}([\mathbb{S}^+_{\mathbb{C}}(N_g) - \mathbb{S}^-_{\mathbb{C}}(N_g)])}$$

Recall the definition of the equivariant Chern character at the end of Section 3.4.3 and that the usual Chern character relates to the $\hat{A}$-form via

$$\hat{A}(E) = (-1)^n\frac{\chi(E)}{ch([\mathbb{S}^+_{\mathbb{C}}(E) - \mathbb{S}^-_{\mathbb{C}}(E)])}$$

for E real spin vector bundle of dimension $2n$ (cf. Proposition 11.24 [26] or (3.23) and the above).

The equivariant $\hat{A}$-form is thus an extension of the usual $\hat{A}$-form but capturing the effective topological behavior of an action. Note that this definition in particular makes sense when regarding sub-manifolds and for charts of orbifolds if g is an element of the corresponding isotropy group; see (2.6) and above. The extension to the inertia orbifold can be understood as capturing the effective symmetries by the isotropy groups of an orbifold within the usual $\hat{A}-$ form.

Definition 5.14. Let $(\psi_{[\gamma]}, V^\gamma/\rho(Z_\Gamma(\gamma))), \gamma \in \Gamma$ be orbifold charts for $\Lambda B^4_{\underline{a},\underline{b}}$ the inertia orbifold in 5.10 corresponding to charts $(\psi, V/\rho(\Gamma))$ as explained in (2.4). Let further $m(\gamma)$ be the multiplicities of $(\psi(0), [\gamma]) \in \Lambda B^4_{\underline{a},\underline{b}}$. Then define $\hat{A}_{\Lambda B^4_{\underline{a},\underline{b}}}(TB^4_{\underline{a},\underline{b}})$ to be the form given by

$$(\psi \circ p)^*(\hat{A}_{\Lambda B^4_{\underline{a},\underline{b}}}(TB^4_{\underline{a},\underline{b}})) = \frac{1}{m(\gamma)}\hat{A}_{\tilde{\gamma}}(TV)$$

where $p : V^\gamma \to V^\gamma/\rho(Z_\Gamma(\gamma))$ is the projection and $\hat{A}_{\tilde{\gamma}}$ as defined in 5.13 for a lift $\tilde{\gamma}$ obtained as in 2.6. Going through all the definitions, one analogously obtains a definition of $\hat{L}_{\Lambda B}(TB^4_{\underline{a},\underline{b}})$.

Remark 5.15. By Definition 5.13, the above does not depend on the choice of a lift $\tilde{\gamma}$ up to a sign but only on its conjugacy class. Further when fixing an orientation on TV_{V^γ} but changing on V^γ changes the sign since $\mathbb{S}^+_{\mathbb{C}}(N_\gamma)$ and $\mathbb{S}^-_{\mathbb{C}}(N_\gamma)$ are swapped in 5.13.

Recall from 3.62 that the spectral functions of the Dirac operators on $M^7_{\underline{a},\underline{b}}$ measure the defect of a coboundary to be closed. They are given by the difference between the $\hat{A}$- genus or L-genus and the index (plus the dimension of the kernel), respectively, the signature. Recall that the $\hat{L}$-class is just the rescaled L-class (cf. 3.51). When considering the adiabatic limits of the spectral functions, i.e., those obtained by shrinking the fibers infinitely small, the following holds.

Theorem 5.16. *[Theorem 1.8 [10]] Assume the assumptions of 5.10 hold and consider the adiabatic family of the Riemannian submersion metric $g_{M^7_{\underline{a},\underline{b}}}$ and the corresponding adiabatic family of Dirac operators. Then*

$$\lim_{\epsilon\to 0}\eta(\mathbb{D}^+_{M^7_{\underline{a},\underline{b}},\epsilon}, 0) = \int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{A}_{\Lambda B}(TB^4_{\underline{a},\underline{b}})2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(\mathbb{D}^+|_{S^3}) \tag{5.14}$$

$$\lim_{\epsilon\to 0}\eta(D^+_{\mathbb{C}\,M^7_{\underline{a},\underline{b}},\epsilon}, 0) = \int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{L}_{\Lambda B}(TB^4_{\underline{a},\underline{b}})2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(D^+_{\mathbb{C}}|_{S^3}) + \lim_{\epsilon\to 0}\tau_\epsilon, \tag{5.15}$$

where the $\eta_{\Lambda B^4_{\underline{a},\underline{b}}}$ are forms on the inertia orbifold defined out of an equivariant η form associated to the spectral function (similar to 5.14, see (1.12) and (1.13) in [10] for a definition) and τ_ϵ is the (finite) sum over the signums of the very small eigenvalues of $D^+_{\mathbb{C}\,M^7_{\underline{a},\underline{b}},\epsilon}$. Since the $\eta_{\Lambda B^4_{\underline{a},\underline{b}}}$ also change sign depending on choosing a lift (and orientation), together with Remark 5.15, the right-hand sides of the formulas do not depend on a choice.

Remark 5.17. Theorem 5.16 results from decomposing the Dirac operators in vertical and horizontal parts (cf. Lemma 2.1 [9]). The original formulas (Theorem 0.1 and Corollary 1.10 in [9]) are formulated for general Seifert fibrations. In this case, equation 5.14 admits additional terms involving some horizontal component and the sum over the signums of the very small eigenvalues. The further assumption of positive scalar curvature of the fibers ensures that these terms vanish (since positive scalar curvature implies by 3.40 that $\mathbb{D}^+|_{S^3}$ is invertible. The claim follows for sufficiently small ϵ by the definition of the terms, cf. Chapter 1.2 [10]). The additional horizontal term in 5.15 vanishes due to the orbifold being even-dimensional (cf. Proposition 2.5 [9]).

Proof of Theorem 5.10. Let $\nabla^{M^7_{\underline{a},\underline{b}},\epsilon}$ be a family of Levi-Civita connections with respect to the adiabatic family $g_{M^7_{\underline{a},\underline{b}},\epsilon}$. Recall the orthonormal frame by (5.13). For each ϵ, we can orthogonally identify it with the frame at $\epsilon = 1$. Thus, there is an orthogonal decomposition

$$\lim_{\epsilon\to 0} \nabla^{M^7_{\underline{a},\underline{b}},\epsilon} = \nabla^{\mathcal{V}} \oplus \pi^* \nabla^{B^4_{\underline{a},\underline{b}}}.$$

Similarly since the Pontryagin forms are cunstructed from the curvature form, which is by the above at the limit block diagonal, one obtains

$$\lim_{\epsilon\to 0} p_1(M^7_{\underline{a},\underline{b}}, g_{M^,_{\underline{a},\underline{b}}\epsilon}) = p_1(\mathcal{V}) + \pi^* p_1(B^4_{\underline{a},\underline{b}}).$$

Note again that by 3.40 $dim(ker(\mathbb{D}^+|_{M^7_{\underline{a},\underline{b}},\epsilon})) = 0$ for sufficiently small ϵ (such that $(M^7_{\underline{a},\underline{b}}, g_{M^7_{\underline{a},\underline{b}},\epsilon})$ has positive scalar curvature). Plugging the above and Theorem 5.16 in Theorem 5.9 we obtain

$$\begin{aligned}
\mu(M^7_{\underline{a},\underline{b}}) &= \lim_{\epsilon\to 0} \mu(M^7_{\underline{a},\underline{b}}) \\
&= (\lim_{\epsilon\to 0} \frac{dim(ker(\mathbb{D}^+|_{(M^7_{\underline{a},\underline{b}},\epsilon)})) + \eta(\mathbb{D}^+|_{(M^7_{\underline{a},\underline{b}},\epsilon)}, 0)}{2} \\
&+ \frac{1}{2^5\cdot 7} \lim_{\epsilon\to 0} \eta(D^+_{\mathbb{C}}|_{(M^7_{\underline{a},\underline{b}},\epsilon)}, 0) \\
&- \frac{1}{2^7\cdot 7} \lim_{\epsilon\to 0} \int_{M^7_{\underline{a},\underline{b}}} p_1(M^7_{\underline{a},\underline{b}}, g_{M^7_{\underline{a},\underline{b}},\epsilon}) \wedge \hat{p_1}(M^7_{\underline{a},\underline{b}}, g_{M^7_{\underline{a},\underline{b}},\epsilon})) \bmod \mathbb{Z} \\
&= (\frac{1}{2} \int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{A}_{\Lambda B}(TB^4_{\underline{a},\underline{b}}) 2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(\mathbb{D}^+|_{S^3}) \\
&+ \frac{1}{2^5\cdot 7}(\int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{L}_{\Lambda B}(TB^4_{\underline{a},\underline{b}}) 2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(D^+_{\mathbb{C}}|_{S^3}) + \lim_{\epsilon\to 0} \tau_\epsilon) \\
&- \frac{1}{2^7\cdot 7} \int_{M^7_{\underline{a},\underline{b}}} (p_1(\mathcal{V}) + \pi^* p_1(B^4_{\underline{a},\underline{b}})) \wedge (\hat{p_1}(\mathcal{V}) + \pi^* \hat{p_1}(B^4_{\underline{a},\underline{b}})) \bmod \mathbb{Z}
\end{aligned}$$

where we could pass the limit inside the integral by dominated convergence since $M^7_{\underline{a},\underline{b}}$ is compact. □

What is left to do now is to construct a Riemannian submersion as in 5.10 and calculate the terms with respect to the metric such that we will retrieve Corollary 5.11. Recall from Section 5.1 that $M^7_{\underline{a},\underline{b}} = P^{10}_{\underline{a},\underline{b}}/(\{1\} \times \Delta S^3)$ and that $P^{10}_{\underline{a},\underline{b}}$

admits a cohomogeneity action of $G := S^3 \times S^3 \times S^3$ with singular isotropy groups $Pin(2)_{\underline{a}}, Pjn(2)_{\underline{b}}$.

Lemma 5.18. *Define* $B^4_{\underline{a},\underline{b}} := P^{10}_{\underline{a},\underline{b}}/(\{1\} \times S^3 \times S^3)$ *then* $\pi : M^7_{\underline{a},\underline{b}} \to B^4_{\underline{a},\underline{b}}$ *is a Seifert fibration for* $(a_1, b_1) \neq (1, 1)$ *with fiber* $S^3 \cong (S^3 \times S^3)/\Delta S^3$.

Proof. We examine the action of $\{1\} \times S^3 \times S^3$ on each orbit of the cohomogeneity one action by G on $P^{10}_{\underline{a},\underline{b}}$ to determine the isotropy groups of $\{1\} \times S^3 \times S^3$. Consider $(1, x_2, x_3) \in \{1\} \times S^3 \times S^3, [q_1, q_2, q_3] \in G/Pin(2)_{\underline{a}}$ then

$$\begin{aligned}
&(1, x_2, x_3)[q_1, q_2, q_3] = [q_1, q_2, q_3] \\
&\Leftrightarrow \exists \alpha \in Pin(2)_{\underline{a}} : (q_1, x_2 q_2, x_3 q_3) = (q_1\alpha_1, q_2\alpha_2, q_3\alpha_3) \\
&\Leftrightarrow \alpha_1 = 1, x_2 = q_2\alpha_2\bar{q}_2, x_3 = q_3\alpha_3\bar{q}_3.
\end{aligned}$$

Recall that $Pin(2)_{\underline{a}} = \{(e^{ia_1\frac{\theta}{2}}, e^{ia_2\frac{\theta}{2}}, e^{ia_3\frac{\theta}{2}}) | \theta \in \mathbb{R}\} \cup \{(je^{ia_1\frac{\theta}{2}}, je^{ia_2\frac{\theta}{2}}, je^{ia_3\frac{\theta}{2}}) | \theta \in \mathbb{R}\}$ so the above implies $e^{ia_1\frac{\theta}{2}} = 1$ for some $\theta \in \mathbb{R}$. Since $e^{i\frac{\theta}{2}} \in S^1$ is an a_1-th root of unity, we get a description of the isotropy group of an element $[q_1, q_2, q_3] \in G/Pin(2)_{\underline{a}}$ by

$$\{(1, q_2\xi^{a_2}\bar{q}_2, q_3\xi^{a_3}\bar{q}_3) | \xi \in S^1 : \xi^{b_1} = 1\} \cong \mathbb{Z}_{|a_1|}.$$

The isomorphism is given by sending the k-th root to k and is injective since $gcd(a_1, a_2, a_3) = 1$. An analogous statement holds for elements in $G/Pjn(2)_{\underline{b}}$. Clearly the action on the principal orbit $G/\Delta Q$ is free since for $\alpha \in \Delta Q$ with $\alpha_1 = 1$ we can directly conclude that $\alpha = 1$. So all together we see that the isotropy groups under the action of $\{1\} \times S^3 \times S^3$ at each point in $P^{10}_{\underline{a},\underline{b}}$ are finite. There is at least one non-trivial isotropy group if and only if $(a_1, b_1) \neq (1, 1)$. Therefore $B^4_{\underline{a},\underline{b}}$ is an orbifold for $(a_1, b_1) \neq (1, 1)$ with singular points being the images of $G/Pin(2)_{\underline{a}}, G/Pjn(2)_{\underline{b}}$. The claim now follows by Remark 2.27. □

To construct a metric on $M^7_{\underline{a},\underline{b}}$ (and $B^4_{\underline{a},\underline{b}}$) fulfilling the assumptions in Corollary 5.10 with π the Seifert fibration from Lemma 5.18 recall from Section 5.2 that $M^7_{\underline{a},\underline{b}}$ decomposes as two disk bundles

$$\begin{aligned}
D^2_\epsilon \to M_- \to (S^3 \times S^3)//Pin(2)_{\underline{a}} \\
D^2_\epsilon \to M_+ \to (S^3 \times S^3)//Pjn(2)_{\underline{b}}
\end{aligned}$$

glued along $(S^3 \times S^3)//\Delta Q$, where $D^2_\epsilon = \{z \in \mathbb{C} | |z| < 1 + \epsilon\}, \epsilon > 0$. Note that we omitted the radius of the disks earlier since it did not make a difference in cohomology computations, whereas we now aim to construct metrics. Our goal is to construct a metric from the pullback of a metric on $(S^3 \times S^3)//\Delta Q \times (-1, 1)$ that satisfies our desired properties.
We have seen (cf. 5.4) that $(G/Pin(2)_{\underline{a}})/(\{1\} \times S^3) \cong (S^3 \times S^3)//Pin(2)_{\underline{a}}$ and therefore that the action of $Pin(2)_{\underline{a}}$ on the normal disk of $M_- \cong (S^3 \times S^3) \times_{Pin(2)_{\underline{a}}} D^2_\epsilon$ is given by the action of $Pin(2)_{\underline{a}}$ on the normal disk of the bundle $G \times_{Pin(2)_{\underline{a}}} D^2$ over $G/Pin(2)_{\underline{a}}$ (cf. 5.3). Moreover, this action is uniquely determined by the action on the boundary, i.e., on $S^1 \cong Pin(2)_{\underline{a}}/\Delta Q$ (there is only one continuous extension to

the interior). Therefore, it has stabilizer ΔQ, so one can conclude that it must be equivalent to

$$Pin(2) \times S^1 \to S^1, (\alpha, z) \to \begin{cases} e^{4i\frac{\theta}{2}} z, & \alpha = e^{i\frac{\theta}{2}} \\ e^{4i\frac{\theta}{2}} \bar{z}, & \alpha = e^{i\frac{\theta}{2}} j. \end{cases} \tag{5.16}$$

Using this description and the $Pin(2)$ equivariant diffeomorphism

$$D^2_\epsilon \backslash \{0\} \to S^1 \times (0, 1+\epsilon), z \to (\frac{z}{\bar{z}}, |z|)$$

we get a diffeomorphism

$$\begin{aligned} \phi_- : (S^3 \times S^3) \times_{Pin(2)_{\underline{a}}} D^2_\epsilon \backslash \{0\} \to & (S^3 \times S^3) \times_{Pin(2)_{\underline{b}}} (S^1 \times (0, 1+\epsilon)) \\ & \cong (S^3 \times S^3) // \Delta Q \times (-1, \epsilon) \\ [q_1, q_2, z] \to & ([q_1, q_2], |z| - 1). \end{aligned}$$

Similarly, we obtain the diffeomorphism

$$\begin{aligned} \phi_+ : (S^3 \times S^3) \times_{Pjn(2)_{\underline{b}}} D^2_\epsilon \backslash \{0\} \to & (S^3 \times S^3) \times_{Pjn(2)_{\underline{b}}} (S^1 \times (0, 1+\epsilon)) \\ & \cong (S^3 \times S^3) // \Delta Q \times (-\epsilon, 1). \end{aligned}$$

Assume $\epsilon \in (0, \frac{1}{4})$ and let $\tau : M^7_{\underline{a},\underline{b}} \to [-1, 1]$ be the projection onto the leave space of the codimension one foliation by the biquotients (note that if ϵ exceeds the interval we need to rescale, as also mentioned in Section 5.2) such that under ϕ_- and ϕ_+ we have $\tau^{-1}([-1, \epsilon)) = M_-, \tau^{-1}((-\epsilon, 1]) = M_+$. The intersection $M_- \cap M_+ = \tau^{-1}(-\epsilon, \epsilon)$, which is a neighborhood of the respective boundaries, is glued via

$$\begin{aligned} \phi_+^{-1} \circ \phi_- : \tau^{-1}(-\epsilon, \epsilon) &\to \tau^{-1}(-\epsilon, \epsilon) \\ [q_1, q_2, z] &\to [q_1, q_2, 2 - z]. \end{aligned}$$

The endpoints of $[-1, 1]$ correspond to the centers of the normal disks, the singular leaves, so putting together ϕ_- and ϕ_+, we obtain a diffeomorphism

$$\phi : M^7_{\underline{a},\underline{b}} \backslash \tau^{-1}(\{-1, 1\}) \to (S^3 \times S^3) // \Delta Q \times (-1, 1)$$

with $\phi|_{M_\pm \backslash \tau^{-1}(\{\pm 1\})} = \phi_\pm$. ϕ describes the foliation of $M^7_{\underline{a},\underline{b}}$ in the sense that $M^7_{\underline{a},\underline{b}}$ decomposes in singular leaves and regular leaves diffeomorphic to $(S^3 \times S^3) // \Delta Q \times (-1, 1)$ (i.e. there are "$(-1, 1)-$times" regular leaves). Note that $t \in (-1, 1)$ is $r \pm 1$ for r describing the radius of a circle in the disc $D^2_\epsilon \backslash \{0\}$ corresponding to $M_\pm$. To construct a metric on $M^7_{\underline{a},\underline{b}}$ one defines on the one hand a smooth metric $g = g_t + dt^2$ on $(S^3 \times S^3) // \Delta Q \times (-1, 1)$ that pulls back under ϕ and on the other hand a smooth metric $\tilde{g} = \tilde{g}_r + dr$ on $S^3 \times S^3 \times D^2_\epsilon$ that projects to metrics $\tilde{g}_\pm$on $M_\pm$. Afterward, one shows that $\phi_{*_\pm}(g)|_{M_\pm}$ coincides with $\tilde{g}_\pm$ for $r \in (0, \epsilon)$, i.e., that $\phi_*(g)$ extends smoothly as t goes to ± 1 to $\tilde{g}_\pm$. One starts by defining a metric $\hat{g} = \hat{g}_t + dt^2$ on $(S^3 \times S^3) \times (-1, 1)$ such that $\hat{g}_t$ is invariant under

the action of ΔQ and thus that $\hat{g}$ induces g (induced submersion metric, cf. Theorem 2.12). To do so, consider the left-invariant basis vector fields of $S^3 \times S^3$ given for each $\alpha \in \{i, j, k\}$ by

$$E_\alpha(q_1, q_2) := \frac{d}{ds}(q_1 exp(s\alpha), q_2)|_{s=0} = (q_1\alpha, 0)$$
$$F_\alpha(q_1, q_2) := \frac{d}{ds}(q_1, q_2 exp(s\alpha))|_{s=0} = (0, q_2\alpha).$$

Define $\langle \cdot, \cdot \rangle$ to be the standard bi-invariant metric on $S^3 \times S^3$ such that the above vector fields are an orthonormal basis. Observe that for $X_\alpha(q_1, q_2) := (0, \alpha q_2)$

$$\frac{\partial}{\partial \theta_+} := b_1 E_i - b_2 X_i + b_3 F_i \tag{5.17}$$
$$\frac{\partial}{\partial \theta_-} := a_1 E_j - a_2 X_j + a_3 F_j \tag{5.18}$$

describe tangents along the action of $Pin(2)_{\underline{a}}$ respectively $Pjn(2)_{\underline{b}}$ on $S^3 \times S^3$ (recall the biquotient action defined above of (5.4)). Now, one can define $\hat{g}_t$ in terms of the metric $\langle \cdot, \cdot \rangle$ and some scaling functions (appearing due to technical reasons, see Section 3 [10] for details)

$$u_\pm, v_\pm, h_\pm : (S^3 \times S^3)//\Delta Q \times (-1, 1) \to \mathbb{R}$$

such that an orthonormal frame for $(M^7_{\underline{a},\underline{b}}, \phi_*(g))$ is given by

$$\bar{e}_0 := \phi_*^{-1}(\frac{\partial}{\partial t}), \qquad \bar{e}_1 := \frac{1}{h_-}(e_i - u_- x_i + v_- f_i)$$
$$\bar{e}_2 := \frac{1}{h_+}(e_j - u_+ x_j + v_+ f_j), \qquad \bar{e}_3 := e_k$$
$$\bar{f}_\alpha := f_\alpha$$

where the lower capitals denote the respective pullbacks of the projected vector fields $E_\alpha, F_\alpha, X_\alpha$ onto the quotient $S^3 \times S^3//\Delta Q$. By the definition, we see that $\bar{e}_0$ is the tangent along the radius of D^2_ϵ, $\bar{e}_1, \bar{e}_2$ describe the tangents along the actions of $Pin(2)_{\underline{a}}, Pjn(2)_{\underline{b}}$ respectively on $S^3 \times S^3$, $\bar{e}_3$ the tangent in k direction of the first component of $S^3 \times S^3$ and the $\bar{f}_\alpha$ the tangents of the second component of $S^3 \times S^3$ which are the ones along the fibers of the Seifert fibration.
For the extension to the singular leaves, one again first defines an $Pin(2)_{\underline{a}}$ respectively $Pjn(2)_{\underline{b}}$ invariant metric $\tilde{g} = \tilde{g}_r + dr^2$ on $S^3 \times S^3 \times D^2_\epsilon$ from which one obtains the induced submersion metrics on the quotients $M_\pm$. Again one uses $\langle \cdot, \cdot \rangle$ and the standard metric on D^2_ϵ to define $\tilde{g}$ in a compatible way, meaning in a way, such that orthonormal frames in a neighborhood of $0 \in D^2_\epsilon$ are given by the $\bar{e}_\alpha, \bar{f}_\alpha$. We refer to Proposition 3.2 in [10] for details. We denote the constructed metric by $g_{M^7_{\underline{a},\underline{b}}}$
Analogously, one obtains a metric $g_{B^4_{\underline{a},\underline{b}}}$ for the orbifold such that away from the singular orbits a local orthonormal frame of vector fields on $B^4_{\underline{a},\underline{b}}$ given by $\check{e}_0, \check{e}_1, \check{e}_2, \check{e}_3$ such that $\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3$ are horizontal lifts and π is a Riemannian submersion (cf. Corollary 3.4 [10]). Furthermore, one can show that $\pi : M^7_{\underline{a},\underline{b}} \to B^4_{\underline{a},\underline{b}}$ with the

metrics described above, has totally geodesic fibers. One needs to prove that for $\nabla^{TM^7_{\underline{a},\underline{b}}}$ the Levi-Civita connection induced by $g_{M^7_{\underline{a},\underline{b}}}$, the horizontal parts of $\nabla^{TM^7_{\underline{a},\underline{b}}}_{\bar{f}_\alpha}\bar{f}_\beta$ are zero, i.e., the second fundamental form of the fibers vanishes. To do so, prove that $g_{M^7_{\underline{a},\underline{b}}}(\nabla^{TM^7_{\underline{a},\underline{b}}}_{\bar{f}_\alpha}\bar{f}_\beta, \bar{e}_\gamma) = 0$. In Lemma 3.6 in [10], this is achieved by Lie bracket calculations and using that due to orthogonality of the frames $g_{M^7_{\underline{a},\underline{b}}}(\nabla^{TM^7_{\underline{a},\underline{b}}}_{\bar{f}_\alpha}\bar{f}_\beta, \bar{e}_\gamma) = -g_{M^7_{\underline{a},\underline{b}}}(\bar{f}_\beta, \nabla^{TM^7_{\underline{a},\underline{b}}}_{\bar{f}_\alpha}\bar{e}_\gamma)$. By construction (out of the standard round metric), the fibers have positive scalar curvature. We are finally able to apply Corollary 5.10. To conclude Corollary 5.11, we are left to elaborate the terms involved in 5.10 with respect to the metrics defined above.

Lemma 5.19. *With respect to the metrics $g_{M^7_{\underline{a},\underline{b}}}$ and $g_{B^4_{\underline{a},\underline{b}}}$ defined above it holds that*

$$4a_1^2b_1^2m^2\frac{1}{a_1^2b_1^2} = \int_{M^7_{\underline{a},\underline{b}}} (p_1(\mathcal{V}) + \pi^*p_1(B^4_{\underline{a},\underline{b}})) \wedge (\hat{p}_1(\mathcal{V}) + \pi^*\hat{p}_1(B^4_{\underline{a},\underline{b}})) \tag{5.19}$$

$$\begin{aligned} -\frac{1}{a_1^2b_1^2} - D(\underline{a}) + D(\underline{b}) =& 2^6\cdot 7\int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{A}_{\Lambda B^4_{\underline{a},\underline{b}}}(TB^4_{\underline{a},\underline{b}})2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(\mathbb{D}^+|_{S^3}) \\ &+ 4\int_{\Lambda B^4_{\underline{a},\underline{b}}} \hat{L}_{\Lambda B^4_{\underline{a},\underline{b}}}(TB^4_{\underline{a},\underline{b}})2\eta_{\Lambda B^4_{\underline{a},\underline{b}}}(D^+_{\mathbb{C}}|_{S^3}) \end{aligned} \tag{5.20}$$

$$\lim_{\epsilon\to 0}\tau_\epsilon = 1 \tag{5.21}$$

for $m, D(\underline{a}), D(\underline{b})$ as in Corollary 5.11.

$p_1(\mathcal{V}) = \frac{1}{8\pi^2}tr((\Omega^{\mathcal{V}})^2)$ and $p_1(B^4_{\underline{a},\underline{b}}) = \frac{1}{8\pi^2}tr((\Omega^{B^4_{\underline{a},\underline{b}}})^2)$ can be computed directly from the definition of the curvature two-forms $\Omega^{\mathcal{V}}, \Omega^{B^4_{\underline{a},\underline{b}}}$(cf. Definition A.17 and for the calculations see Lemma 3.9 and 3.12 in [10]). For example the connection one form of the vertical bundle with respect to the Levi-Civita connection $\nabla^{\mathcal{V}}$ is given by the entries $g_{M^7_{\underline{a},\underline{b}}}(\bar{f}_\alpha, \nabla^{\mathcal{V}}\bar{f}_\beta)_{\alpha,\beta}$ (cf. beginning of Section 3.4.1). The computations of $\hat{p}_1(\mathcal{V})$ and $\pi^*\hat{p}_1(B^4_{\underline{a},\underline{b}})$ are a little harder. First of all one needs to find a primitive for $p_1(\mathcal{V}) + \pi^*p_1(B^4_{\underline{a},\underline{b}})$. The idea is to find primitives $\kappa_\pm$ on $M_\pm$ and closed three-forms $\nu_\pm$ such that $(\kappa_- + \nu_-) - (\kappa_+ + \nu_+) = 0$ on $\tau^{-1}(-\epsilon, \epsilon)$. To find the $\kappa_\pm$ one considers the vector orbi bundle $W \to B^4_{\underline{a},\underline{b}}$ of rank four, i.e. $W = P^{10}_{\underline{a},\underline{b}} \times_{S^3\times S^3} \mathbb{H}$, for which the Pontryagin form can be again directly calulated (cf. Lemma 3.11 [10]) and a primitive is easier to find. Since $\mathcal{V}$ and π^*W are stably isomorphic (π is the associated unit-sphere bundle) one finds forms $\xi_\pm$ such that $p_1(\mathcal{V})|_{M_\pm} = \pi^*p_1(W)|_{M_\pm} + d\xi_\pm$. Therefore one finds primitives $\kappa_\pm$ by solving

$$d\xi_\pm + \pi^*p_1(W)|_{M_\pm} + \pi^*p_1(B^4_{\underline{a},\underline{b}})|_{M_\pm} = d\kappa_\pm$$

(cf. Lemma 3.15 [10]). Afterward, the $\eta_\pm$ can be found (cf. Lemma 3.16 [10]) and the combined result (5.19) is proven in Theorem 3.18 in [10].
To see (5.20) some further thoughts on the inertia orbifold $\Lambda B^4_{\underline{a},\underline{b}}$ are necessary. In fact, it decomposes into three connected components

$$\Lambda B^4_{\underline{a},\underline{b}} = B^4_{\underline{a},\underline{b}} \sqcup (S^2_- \times \{1, ..., \frac{|a_1|-1}{2}\}) \sqcup (S^2_+ \times \{1, ..., \frac{|b_1|-1}{2}\}). \tag{5.22}$$

It is clear that $B^4_{\underline{a},\underline{b}} \subset \Lambda B^4_{\underline{a},\underline{b}}$ is the component consisting of the fixed points by the identity element. Recall from the proof of Lemma 5.18 that the singular sets of $B^4_{\underline{a},\underline{b}}$ are the images of $G/Pin(2)_{\underline{a}}, G/Pjn(2)_{\underline{b}}$. Since the action of $\{1\} \times S^3 \times S^3$ commutes with the action of $Pin(2)_{\underline{a}}$ respectively $Pjn(2)_{\underline{b}}$ on G, the images are given by $S^3/Pin(2)_{\underline{a_1}}$ respectively $S^3/Pin(2)_{\underline{b_1}}$, i.e. by two copies $\mathbb{R}\mathrm{P}^2_{\pm}$ of $\mathbb{R}\mathrm{P}^2$. Denote by $S^2_{\pm}$ their respective double covers. Further recall that the isotropy groups for the singular sets are given by $\mathbb{Z}_{|a_1|}$ and $\mathbb{Z}_{|b_1|}$ respectively, with generators $\gamma_{\pm}$. The claim of 5.22 is now, that $(b, [\gamma^s_{\pm}])$ and $(b, [\gamma^{s^{-1}}_{\pm}])$ are path connected in $S^2_{\pm} \times \{s\}$ for each $b \in B^4_{\underline{a},\underline{b}}, s \in \{1, ..., \frac{|c|-1}{2}\}, c = a_1$ or b_1 . For a proof of such a curve, we refer to Corollary 2.4 in [10]. Now, one makes use of the Decomposition (5.22) by splitting the integrals on the right-hand side of (5.20) in a similar fashion. The integrals over $B^4_{\underline{a},\underline{b}}$ give the $\frac{1}{a_1^2 b_1^2}$ term (cf. Lemma 3.11,3.13 in [10] and Proposition 4.2 in cf. [9]). The other integrals are computed by evaluating the definition of the forms (see Theorem 3.19 [10] for the calculations) and give the Dedekind sums. The last equation (5.21) follows from the observation that τ_ϵ is the signature of a quadratic form on the fourth page of the Serre Spectral sequence corresponding to the Seifert fibration (see (1.14) in [10] for the quadratic form and note by comparing the definition of τ_ϵ the abuse of notation). For the computation, see Theorem 3.21 in [10].

Lemma 5.19 together with Corollary 5.10 now imply Corollary 5.11. Further computations show (cf. Chapter 6 in [8] and Example 3.20 in [10]) that for the homotopy spheres $M^7_{\underline{a},\underline{b}}$

$$\mu(M^7_{\underline{a},\underline{b}}) = \frac{1 - a_1^2 b_1^2 m^2}{2^5 \cdot 7} - \frac{4l+1}{28} \bmod \mathbb{Z}$$

for $\underline{a} = (-3, -3, 1), \underline{b} = (1, 4r+1, 4r+1)$. The non-Milnor spheres are thus attained for $r \in \{-3, -1, 1, 2, 4, 8, 11, 15\}$. Recall from Chapter 4 that the Milnor spheres appear at $(a_1, b_1) = (1, 1)$.

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Appendix A

Fiber Bundles

Unless specified, this chapter is based on [19]. We aim to give a brief introduction to principal bundles and associated bundles, along with some essential statements for this thesis.

Definition A.1. Let G be a topological group. A right $G-$space is a space X with an action $X \times G \to X, (x, g) \to xg$, such that

1. $\forall x \in X, s, t \in G : x(st) = (xs)t$.
2. $\forall x \in X : xe = x$.

One defines a left $G-$space analogously.

Remark A.2. If X is a left $G-$space then $(x, g) \to xg := g^{-1}x$ defines a right $G-$space.

Remark A.3. A bundle is a triple consisting of a base space B, a total space X and a map $p : X \to B$. $p^{-1}(b)$ is called the fiber of $b \in B$. For every $G-$space X, one obtains a bundle $\rho = (X, p, X/G)$, $p : X \to X/G, p$ quotient map.

Definition A.4. Let X be a free G- space (e.g., the action is free). A principal $G-$bundle is a bundle (X, p, B) such that

1. there exists a homeomorphism $X/G \to B$ and
2. there exists a continuous translation function. That is a function $\tau : X^* \to G$ such that $x\tau(x, x') = x'$, where $X^* := \{(x, xg) \in X \times X | x \in X, g \in G\}$.

A principal G-bundle has fiber G. Condition 2 guarantees that the action of G on the fibers is continuous, transitive, and free. Continuity and transitivity directly follow from the definition, and freeness is due to $x = xg \implies x = x\tau(x, xg) = x\tau(x, x)$. Since τ is a well defined function it must hold that $\tau(x, xg) = \tau(x, x) = e$. Notice, that together with 1 this also implies local triviality, i.e. there is an open covering $\bigcup_{i \in I} U_i$ of B such that $p^{-1}(U_i)$ is homeomorphic to $U_i \times G$. On the other hand, these trivializations, together with the information on how intersections of the U_i are glued together with G ($\tau(x, x')$ for $x, x' \in U_i \cap U_j$) determine the principal $G-$bundle.

Out of principal bundles, we can construct bundles with the same local trivializations (i.e., the same covering $\bigcup_{i\in I} U_i$) but different fibers (i.e., $p^{-1}(U_i)$ is homeomorphic to $U_i \times F$ for a left $G-$ space F). Such bundles are called fiber bundles. Of course, principal bundles are also fiber bundles.

Definition A.5. Let $\rho = (X, p, B)$ be a principal $G-$bundle and $\varphi : G \to Homeo(F)$ a continuous left G action. One can define a right $G-$space structure on $X \times F$ by $(x, f)g := (xg, \varphi(g)^{-1}f)$. Let $X \times_G F := (X \times F)/G$, $p_F : X \times_G F \to B$ the composition $pr_x \circ p : X \times F \to B$ factored by the action of G. Then $(X \times_G F, p_F, B)$ is called the fiber bundle over B with fiber F and structure group G associated to ρ. To emphasize the representation φ one often writes $X \times_\varphi F$ instead of $X \times_G F$.

Remark A.6. The name structure group explains itself by recognizing that $(X \times_G F,\ p_F, B)$ arises from copies of F for each $b \in B$ glued together by the action of G on X and F. Notice that $(x, f) \sim (xg, \varphi(g)^{-1}f)$ implies $(xg, f) \sim (x, \varphi(g)f)$, $\forall g \in G$. This ensures that the principal $G-$bundle and its associated bundles share the same gluing information. In particular, every fiber bundle can be described by local trivializations, as in the case of principal $G-$bundles.

Definition A.7. A section of a fiber bundle $\rho = (X, p, B)$ is an open set $U \subseteq B$ together with a map $s : U \to X$, such that $p \circ s = id$. We denote by $\Gamma(X)$ the space of smooth sections of X.

Definition A.8. Let $\rho_X = (X, p_X, B), \rho_Y = (Y, p_Y, B)$ be two fiber bundles over the same base space B. A bundle map is a continuous map $\varphi : X \to Y$ such that $p_Y \circ \varphi = p_X$. If ρ_X and ρ_X are principal $G-$bundles, we say that they are equivalent if $\varphi(xg) = \varphi(x)g, \forall g \in G$. If we say that two principal $G-$bundles are the same, we mean that they are equivalent.

Remark A.9. Using the fact that principal bundles are determined by local trivializations and the gluing information (that fulfill the cocycle condition), one can show that the equivalence classes of principal $G-$bundles $Prin_G(X)$ are $H^1(X; G)$ (see Appendix A in [26]).

Theorem A.10. *Let $\rho = (X, p_G, B)$ a principal G-bundle, H a subgroup of G, and $\nu = (Y, p_H, B)$ the principal H-bundle obtained as the restriction of ρ. If F is a left G and H space, there is a natural isomorphism $X \times_G F \to Y \times_H F$.*

Definition A.11. Let $\rho = (X, p, B)$ be a fiber bundle and $f : B' \to B$ a map. Define X' as the subspace of pairs $(b', x) \in B' \times X$ with $f(b') = p(x)$ and $p' : X' \to B'$ as sending (b', x) to b'. Then the pullback bundle by f is defined as the bundle $f^*(\rho) := (X', p', B')$, i.e. the fibers are $p'^{-1}(b') = \{(b', x) | x \in p^{-1}(f(b'))\}$.

Definition A.12. A fiber bundle $\rho = (X, p, B)$ with fiber F is called an $n-$dimensional vector bundle, if each fiber $p^{-1}(x)$ is an $n-$dimensional vector space over a field k and the structure group G is a subgroup of $GL_n(k)$. One can do a reverse construction to the fiber bundle construction A.5 on vector bundles to obtain the associated principal $GL_n(k)-$bundle. This is the so-called frame bundle.

A frame at $b \in B$ is an ordered basis of $p^{-1}(b)$ (which is a vector space, so $GL_n(k)$ acts on it). Denote by F_b the set of all frames at b, then $E := \bigsqcup_{b \in B} F_b$ is the frame bundle corresponding to X. One can show that the associated fiber bundle to E with fiber F is exactly ρ, i.e. $X = E \times_\varphi F$, where $\varphi : g \to Homeo(F)$.

An orientation of a vector bundle naturally extends the definition of orientation of vector spaces.

Definition A.13. A vector bundle is orientable if one can choose orthonormal frames for each base point such that the structure group of the resulting frame bundle consists only of elements with positive determinant.

Definition A.14. A Riemannian vector bundle is a vector bundle with a smooth assignment for each point of the base space to a scalar product on the fiber. Analogously, one defines a Hermitian vector bundle as a holomorphic vector bundle equipped with a smoothly varying Hermitian inner product on its fibers.

Example A.15. The tangent bundle to a Riemannian manifold (M, g) of dimension n is a Riemannian vector bundle. One defines local trivializations by the local charts $(U_i, \varphi_i)_{i \in I}$ of (M, g). For $v \in T_pM, p \in U_i$ we can define

$$p^{-1}(U_i) \to U_i \times \mathbb{R}^n, v \to (p, (v_1, ..., v_n)),$$

where the v_j are the components of v with respect to φ_i.
Let $\tilde{M} \subset M$ be a submanifold. The normal bundle $N\tilde{M}$ to $\tilde{M}$ in M is defined to be the vector space such that $TM|_{\tilde{M}} = T\tilde{M} \oplus N\tilde{M}$. Analogously to above one can see that $N\tilde{M}$ is a Riemannian vector bundle over $\tilde{M}$.

As for Riemannian manifolds, one can parallel-transport along principal G-bundles over a manifold M, where G is a Lie group. They analogously describe how to parallel-transport along the bundle, i.e., how to connect the fibers. The following is based on Chapter 2.4 [26].

Definition and Remark A.16. Let G be a Lie group, M a manifold of dimension n and $\rho = (X, p, M)$ a principal $G-$bundle. Denote by $\mathfrak{g}$ the Lie algebra of G. The vertical tangent space $\mathcal{V}_x$ at a point $x \in X$ is the tangent space to the orbit through x. One obtains an isomorphism

$$\gamma_x : \mathfrak{g} \to \mathcal{V}_x \cong T_x(Gx), V \to \tilde{V}_x := \frac{d}{dt}(xexp(tV))|_{t=0}.$$

$\tilde{V}_x$ is indeed in the vertical tangent space of x since $xexp(tV)$ describes the action of $exp(tV) \in G$ on x. For more details, see Chapter 20 in [28]. A connection on X is now a $G-$invariant field of $n-$planes τ in the tangent space of X such that $\pi_* : \tau_x \to T_{p(x)}M$ is an isomorphism for every $x \in X$. We call τ horizontal since it is complementary to the tangents along the orbit.

An analogous definition of vertical and horizontal spaces is given in Section 2.2 in terms of submersions.

Definition A.17. Let ρ, G be as in A.16. A connection 1-form ω on X is given by

$$\omega_x : T_x X \to \mathfrak{g}, \forall x \in X$$

defined by the isomorphism γ_x and the projection $T_x(X) \to \mathcal{V}_x$. The curvature of the connection is the 2-form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ for $[,]$ the usual Lie algebra bracket.

Remark A.18. The following properties hold:

1. $\tau_x = ker(\omega_x)$.
2. $\omega(\tilde{V}) \equiv V, \quad \forall V \in \mathfrak{g}$.
3. $\Omega(\tilde{V}, \cdot) \equiv 0, \quad \forall V \in \mathfrak{g}$.

Connections provide a way to define parallelism on principal bundles. A tangent vector v at x is parallel, if $\omega_x(v) = 0$. Principal connections also induce connections on any associated bundle (cf. Section 3.4.1).

Proposition A.19 (Proposition 3.1 and following [24])**.** *Let $\rho = (X, p, M)$ be a principal G-bundle, G a Lie group and M a manifold. For a smooth curve $\gamma : [0,1] \to M$ and a point $x_0 \in X$ with $p(x_0) = \gamma(0)$ there exists a unique horizontal lift $\tilde{\gamma} : [0,1] \to X$ of γ (i.e. $p \circ \tilde{\gamma} = \gamma$) which starts from x_0.*

Appendix B

Algebraic Topology

This chapter serves as a reference for several theorems on Cohomology, Homotopy Theory, Covering spaces, and Serre Spectral Sequences. The following is based on [15] and [16]. Let G be an abelian group and R a ring.

B.1 Cohomology

Theorem B.1 (Poincaré Duality with $\mathbb{Z}$ coefficients, Theorem 3.30 [15])**.** *Let M be an orientable, closed n-dimensional manifold. Then one has for every k an isomorphism*

$$H^k(M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z}).$$

Theorem B.2 (Universal Coefficient Theorem, Theorem 3.2 [15])**.** *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C_n, G)$ are determined by split exact sequences*

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C;G) \to Hom(H_n(C), G) \to 0.$$

Corollary B.3. *The Universal Coefficient Theorem implies that whenever $H_{n-1}(C)$ is free abelian and $H_n(C)$ is free abelian and finitely generated then*

$$H^n(C;\mathbb{Z}) \cong H_n(C).$$

Theorem B.4 (Meyer-Vietoris Sequence, pp.203-204 [15])**.** *Let X be a space such that it is the union of the interiors of two spaces A, B. Then there exists a long exact sequence*

$$\cdots \to H^i(X;G) \xrightarrow{\Psi} H^i(A;G) \oplus H^i(B;G) \xrightarrow{\Phi} H^i(A \cap B;G) \to H^{i+1}(X;G) \to \cdots$$

associated to the short exact sequence of cochain complexes

$$0 \to C^n(A+B;G) \xrightarrow{\psi} C^n(A;G) \oplus C^n(B;G) \xrightarrow{\phi} C^n(A \cap B;G) \to 0$$

where the restrictions to A and B are the coordinates of ψ and ϕ takes their difference.

Theorem B.5 (Long exact sequence of a pair, cf. pp.199 [15]). *Let X be a space and A a subspace of X. There exists a long exact sequence of the pair (X, A)*

$$\cdots \to H^i(X, A; G) \to H^i(X; G) \to H^i(A; G) \to H^{i+1}(X, A; G) \to \cdots$$

Theorem B.6 (Künneth formula, 3.15 [15]). *Let X, Y be spaces, then*

$$H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$$

is an isomorphism of rings.

B.2 Homotopy Theory

Definition B.7. A manifold is n-connected if its homotopy groups are trivial up to degree n.

Theorem B.8 (Hurewicz Theorem, Theorem 4.32 [15]). *If a space X is $(n-1)$-connected, $n \geq 2$, then $\pi_n(X) \cong H_n(X)$.*

Theorem B.9 (Whitehead's Theorem, Corollary 4.33 [15]). *A map $f : X \to Y$ between simply-connected CW-complexes is an homotopy equivalence if $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

Definition B.10. A map $p : E \to B$ has the homotopy lifting property with respect to a space X if for a homotopy $H_t : X \to B$ and a lift $\tilde{H}_0$ of H_0 ($p \circ \tilde{H}_0 = H_0$) there exists a lift $\tilde{H}_t : X \to E$ for the whole homotopy H_t. If $\tilde{H}_t$ extends a lift $\tilde{G} : A \to E$ for $A \subseteq X$, then p is said to have the homotopy lifting property of a pair (X, A).

Proposition B.11 (Proposition 4.48 [15]). *A fiber bundle has the homotopy lifting property for all pairs of CW-complexes.*

Theorem B.12. *Let $F \to E \to M$ be a fiber bundle. Then there exists a long exact sequence of homotopy groups*

$$... \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \pi_{i-1}(B) \to ...$$

Proof. With Proposition B.11, fiber bundles have the homotopy lifting property with respect to disks, and thus a long exact sequence of homotopy groups exists (Theorem 4.41 [15]). □

B.3 Covering Spaces

Definition B.13. Let X be a topological space. A covering of X is a topological space $\tilde{X}$ with a map $p : \tilde{X} \to X$ such that

$$\forall x \in X \, \exists U \subseteq X \text{ open with } x \in U : p^{-1}(U) = \bigsqcup_{i \in I} U_i, U_i \subseteq \tilde{X} \text{ open with } p(U_i) \cong U.$$

This indicates that, for every element x in X, there exists a neighborhood U whose preimage is a disjoint union of copies of U. Therefore, U is called evenly covered, and the U_i are called sheets. The number of sheets for $x \in U$ is the cardinality of $p^{-1}(U)$.

Remark B.14. A covering space whose fibers have the same cardinality is a fiber bundle (A.5) with a discrete fiber. On the other hand, a fiber bundle with a discrete fiber is a covering space.

Definition B.15. Two coverings $p : Y \to X, q : Z \to X$ are equivalent, if there exists a homeomorphism $f : Y \to Z$ such that $q \circ f = p$.

Proposition B.16 (Proposition 1.33 [15])**.** *Let $p : (\tilde{X}, \tilde{x}) \to (X, x)$ be a covering and (Y, y) a path and locally path-connected space. Then for any map $f : (Y, y) \to (X, x)$ there exists a lift $\tilde{f} : (Y, y) \to (\tilde{X}, \tilde{x})$, i.e. such that $p \circ \tilde{f} = f$, if and only if $f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}))$.*

Proposition B.17 (Proposition 1.32 [15])**.** *Let $X, \tilde{X}$ be path-connected spaces and $p : \tilde{X} \to X$ a covering. Then for $x \in X, \tilde{x} := p^{-1}(x)$ the number of sheets of x equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}))$ in $\pi_1(X, x)$.*

Theorem B.18 (Theorem 1.38 [15])**.** *Let X be path-connected, locally path-connected, and have the property that for every $x \in X$ there is a neighborhood U of x such that $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion is trivial. Then, there is a bijection between the isomorphism classes of path-connected covering spaces of X and the conjugacy classes of subgroups of $\pi_1(X, x)$.*

Together with Proposition B.17 this implies that the number of $n-$ sheeted, connected coverings of a space X is in one-to-one correspondence to subgroups of index n in $\pi_1(X)$.

Definition B.19. A simply connected covering space of a path-connected, locally path-connected space X is called a universal cover. This is because such a universal cover is also a covering for every other path-connected covering space of X (cf. Section 1.3 [15]).

B.4 Serre Spectral Sequence

Definition B.20. A filtration of a space X is a sequence of subspaces $... \subset X_p \subset X_{p+1} \subset ...$ such that X is the union of the X_p.

The following constructions can be done for any exact triangle (i.e., a diagram that is exact on the corners). Still, for the sake of this paper, we restrict ourselves to the case of Cohomology groups. Let $... \subset X_p \subset X_{p+1} \subset ...$ be a filtration of a space X (e.g. the X_p are the $p-$skeletons) and G a group. Consider the exact triangle B.1 obtained from the long exact cohomology sequence of pairs.
Define $d_1 := j \circ k$, which yields a differential map since $d_1^2 = (j \circ k) \circ (j \circ k) = 0$, since kj is the coboundary map. Further define

$$E_1 := ker(d_1)/im(d_1), \qquad A_1 := i(A) \subset A,$$
$$i_1 := i|_{A_1}, \qquad j_1(i(a)) := [j(a)] \in E_1, \quad k_1([e]) := k(e).$$

One can verify that these definitions are well-defined and yield another exact triangle. This holds true if one continues this construction iteratively.

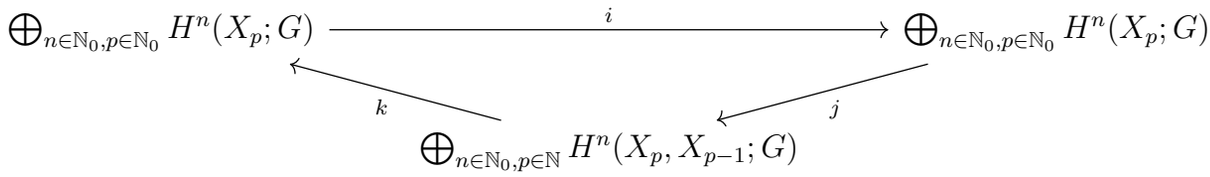


Figure B.1: Exact triple in cohomology

Definition B.21. The sequence $\{E_r, d_r\}$ with

$$d_r \ : \ E_r \ \to \ E_r, E_{r+1} \ = \ ker(d_r)/im(d_r), d_r^2 = 0$$

obtained by the construction above is called a Serre spectral sequence.

This definition holds in the generality of exact triangles. For Cohomology (and, e.g., also for Homology), one obtains a Serre spectral sequence for fibrations. Denote by $E_r^{p,q}$ the $H^p(X_q, X_{q-1})$ part in E_r. We call all elements $E_r^{p,q}$ with their differentials d_r for a fixed r the r-page. A common convention is $X_q = \emptyset$ for $q < 0$.

Theorem B.22. *Let $F \to X \to B$ be a fibration with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F;G)$. Then there exists a spectral sequence $\{E_r^{p,q}, d_r\}$ with the following properties.*

1. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad E_{r+1}^{p,q} = ker(d_r)/im(d_r)$.
2. *There are stable items $E_\infty^{p,n-p} \cong F_p^n/F_{p+1}^n$ in a filtration $0 \subset F_n^n \subset ... \subset F_0^n = H^n(X;G)$. Stable item means here that for fixed p, q the sequence $E_r^{p,q}$ gets stationary for large enough r.*
3. $E_2^{p,q} \cong H^p(B; H^q(F;G))$.

Appendix C

Characteristic Classes

This chapter is based on [31], Appendix B in [26], and Chapter 3 in [17]. We assume in this section that G is a Lie group, that all spaces have base points and are homotopy equivalent to a countable CW complex, and that all maps are base-point-preserving. We provide a brief introduction to classifying spaces to give an intuition for the concept of characteristic classes, which we define at the end.
Given G and a compact Hausdorff space B, one is interested in classifying all possible (equivalence classes of) principal G-bundles over B, denoted by $Princ_G(B)$. This can be done by considering a universal G-bundle, defined as follows.

Definition C.1. Let BG be a connected space and $[B, BG]$ the set of all homotopy classes of maps from B to BG. BG together with a principal G-bundle $EG \to BG$ is the classifying space for G if the map

$$[B, BG] \to Princ_G(B)$$

defined by sending $f : X \to BG$ to $f^*(EG)$ is a bijection. $EG \to BG$ is then called the universal principal G-bundle.

Cf. [31] that such classifying spaces exist.

Theorem C.2. *For principal $G-$bundles over a sphere, one has*

$$Princ_G(S^n) \cong \pi_n(BG) \cong \pi_{n-1}(G).$$

Theorem and Definition C.3. Let R be a ring and BG a classifying space. Fix a non-zero element $c_k \in H^k(BG; R)$, a so-called universal characteristic class. The sum $c := 1 + c_1 + c_2 + ... \in H^*(BG; R)$ is called the total universal characteristic class. One has that for a principal G-bundle $E \to B$ and the corresponding map $f : B \to BG$ the c_k-characteristic class of E $c_k(E) := f^*(c_k) \in H^k(B; R)$ is unique.

Remark C.4. The last statement in C.3 implies that characteristic classes of a trivial bundle are zero (or one if considering the total class), since those are pullbacks of one-point spaces.

For an n-dimensional manifold M, $c_k(M)$ is defined as the characteristic class of the frame bundle to the tangent bundle TM, which is a principal GL_n-bundle. In

general, when addressing characteristic classes of vector bundles, one considers the classes of the frame bundle.

There are two properties of characteristic classes that are important for a consistent application.

1. The Whitney product formula: For two vector bundles E, E' over the same base space X it holds that

$$c(E \oplus E') = c(E) \smile c(E') \in H^*(B; R)$$

and

$$c_k(E \oplus E') = \sum_{i=0}^{k} c_i(E) \smile c_{k-i}(E') \in H^k(B; R).$$

2. Naturality: Let B, B' base spaces of vector bundles E, E'. If $f : B \to B'$ is a map, compatible with a bundle map between E and E', then $c_k(f^*(E)) = c_k(E) = f^*(c_k(E'))$ where f^* denotes the pullback.

We introduce the most important characteristic classes. Note that those are, in particular, homeomorphism invariants.

Theorem and Definition C.5. The ring $H^*(BO_n; \mathbb{Z}_2)$ is the $\mathbb{Z}_2$ polynomial ring $\mathbb{Z}_2[w_1, w_2, ..., w_n]$ on generators $w_k \in H^k(BO_n; \mathbb{Z}_2)$ - the universal k-th Stifel-Whitney classes. They satisfy the Whitney product formula and naturality.

Therefore, Stiefel-Whitney classes are defined for real vector bundles. In 3.2 we discuss an equivalent definition and interpretation of the first two Stiefel-Whitney classes (cf. pp. 81 [26]). But also in higher degrees $n - k + 1$, for an n-dimensional vector bundle, they can be interpreted as the obstruction to finding k orthonormal sections. So they indicate the triviality of a vector bundle or the "twisted-ness". Conversely, vanishing Stiefel-Whitney classes do not guarantee that the bundle is trivial. Chern classes form the analog of Stiefel-Whitney classes for complex vector bundles.

Theorem and Definition C.6. The ring $H^*(BU_n; \mathbb{Z})$ is a $\mathbb{Z}$- polynomial ring $\mathbb{Z}[c_1, ..., c_n]$ on generators $c_k \in H^{2k}(BU_n; \mathbb{Z})$ - the universal k-th Chern classes. The Whitney product formula and naturality hold for the Chern classes.

Proposition C.7. *If one regards a complex n-dimensional vector bundle $E \to B$ as a real $2n-$dimensional vector bundle, then $w_{2i+1}(E) = 0$ and $w_{2i}(E)$ equals $c_i(E)$ under the coefficient map $H^{2i}(B; \mathbb{Z}) \to H^{2i}(B; \mathbb{Z}_2)$ for each $i \in [0, n]$.*

Considering the complexification $E^{\mathbb{C}}$ of a real vector bundle E, $(-1)^k c_{2k}(E^{\mathbb{C}}) \in H^{4k}(B; \mathbb{Z})$ defines another invariant, the Pontryagin class. Since $H^*(BO_n; \mathbb{Z})$ is not a polynomial ring, but has torsion, one often uses the following definition of rational Pontryagin classes, which is equivalent when regarding the image of the integer Pontryagin classes in $H^{4k}(B; \mathbb{Q})$.

Theorem and Definition C.8. Let R be an integral domain containing $\frac{1}{2}$. The ring $H^*(BO_n; R)$ is the polynomial algebra $R[p_1, ..., p_{[\frac{n}{2}]}]$ on generators $p_k \in H^{4k}(BO_n; R)$. For $R = \mathbb{Q}$ they are called the universal k-th rational Pontryagin classes. They fulfill the Whitney product formula.

Theorem and Definition C.9. Let R be as in C.8. For each n, we have the following ring homomorphisms

$$H^*(BSO_{2n+1}; R) = R[p_1, ..., p_n]$$
$$H^*(BSO_{2n}; R) = R[p_1, ..., p_n, \chi]/\langle \chi^2 - p_n \rangle$$

where p_k are the images of the classes from C.8 under $BSO_n \to BO_n$. χ is called the universal Euler class and fulfills for oriented vector bundles the Whitney product formula as well as naturality.

Again, one can define the Euler class as an integral cohomology class. Then (after pulling back, or under mod two reduction respectively), it coincides with the top Chern and Stiefel-Whitney classes.

Notation Index

$[M]$	The fundamental class of a compact, connected, oriented manifold M. This is the generator of the top cohomology.
$*$	Hode-star operator, 3.44
$\bigwedge^* V$	$\bigoplus_{k=0}^{dim(V)} \bigwedge^k V$
$\bigwedge^k V$	k-th exterior product of V
$\mathbb{C}l_n$	$Cl_n \otimes \mathbb{C}$
$\chi(E)$	Euler class of a vector bundle E, C.9
ΔG	The diagonal of a group G, e.g. elements of the form $(g, g), g \in G$
Δ	Laplace operator, 3.46
$\Gamma(E)$	Space of sections of a bundle E
$\hat{A}(M)$	$\hat{A}(TM)[M]$, the $\hat{A}$- genus or form of a manifold M, 3.49
$\hat{A}_{\Lambda B}(TB)$	Inertia $\hat{A}$-form to an orbifold B, 5.14
$\hat{L}(M)$	$\hat{L}(TM)[M]$, the $\hat{L}$- genus or form of a manifold M, 3.51
$\hat{L}_{\Lambda B}(TB)$	Inertia $\hat{L}$-form to an orbifold B, 5.14
ΛB	Inertia orbifold to an orbifold B, 2.28
$\mathbb{D}$	Dirac operator on the unique complex, irreducible spinor bundle to a manifold, 3.42
$\mathbb{S}_{\mathbb{C}}(E)$	Unique complex,irreducible spinor bundle associated to E
$\mathcal{H}$	$\bigoplus_{p=0}^{n} \mathcal{H}^p$, 3.46
$\mathcal{H}^p$	Harmonic $p-$forms on a manifold, 3.46
$\mathcal{H}_p$	Horizontal subspace of a Riemannian submersion at a point p, 2.11
$\mathcal{V}_p$	Vertical subspace of a Riemannian submersion at a point p, 2.11
μ	Eells-Kuiper invariant, 5.7

∇	Covariant derivative, 3.32
$\Omega^*(M)$	$\oplus_{p=0}^{n}\Omega^p(M)$
Ω^E	Curvature form corresponding to a bundle E with a connection, A.17 and following
$\Omega^p(M)$	Smooth p-forms of the manifold M
Ω_n^{Spin}	Spin cobordism ring of dimension n, 3.24
$\pi_n(M)$	n-th homotopy group of a manifold M.
$\sigma_\xi(D)$	Principal symbol of the operator D corresponding to ξ, 3.14
θ_n	Group of homotopy n-spheres, 1.8
A^t	Transpose of the matrix A
$Aut(O)$	The automorphism group of an object O
BG	Classifying space of G, C.1
$C^\infty(X)$	Smooth functions of a space X
$c_k(E)$	k-th Chern class of a vector bundle E, C.6
$ch(E)$	Chern character of a bundle E, 3.58
$Cl(E)$ / $\mathbb{C}l(E)$	Real / Complex Clifford bundle associated to the bundle E, 3.26
$Cl(M)$/ $\mathbb{C}l(M)$	$Cl(TM)$/ $\mathbb{C}l(TM)$ for M manifold
$Cl(V,q)$	Clifford algebra to V and q, 3.1
Cl_n	$Cl(\mathbb{R}^n,q), q$ standard Euclidean form
D	Dirac operator, 3.35
$D^\pm$	Parts of the splitting of the Dirac operator D, 3.37
$D_\mathbb{C}$	Dirac operator on the complex Clifford bundle to a manifold, 3.43
$Der(A)$	The group of derivations on an algebra A
$G//U$	Biquotient of G compact Lie group by U closed subgroup
$G_n(V,q)$	The group of G-transformations on V with respect to the quadratic form q, where, G linear group, $n = dim(V)$
G_x	Isotropy group of x in G
GL_n	General linear group over the reals
Gx	Orbit of x under G.

$ind(D)$	Index of the operator D, 3.41
k	Field of characteristic unequal two
$K(x,y)$	Sectional curvature of the plane spanned by the vectors x, y, 1.1
$L(M)$	$L(TM)[M]$, the L- genus of a manifold M, 3.51
M	Complete, smooth, connected manifold
$m(\gamma)$	Multiplicity of an element γ in an isotropy group of an orbifold, (2.5)
$M_n(k)$	$n \times n$ matrix algebra over a field k
$N(X)$	The normal tangent space of $X \subset M$ for some manifold M and submanifold X
O_n	Orthogonal group over the reals
$P_G(E)$	Bundle of G-frames associated to E, G linear group
$p_k(E)$	k-th rational Pontryagin class of a vector bundle E, C.8
$p_k(M)$	k-th Pontryagin form or class of the tangent bundle of the manifold M
q	Quadratic form
R	Curvature tensor
$S(E)/S_{\mathbb{C}}(E)$	Real / Complex spinor bundle associated to E, 3.29
$S(O(n)O(m))$	A matrix in $SO(n+m)$ consisting of two block matrices in $O(n)$ and $O(m)$
S^n	Standard $n-$dimensional sphere.
$sec(\sigma)$	Sectional curvature of the planeσ, 1.1
$sig(M)$	Signature of the rational quadratic form $\alpha \rightarrow \alpha \smile \alpha \in H^{dim(M)}(M;\mathbb{Q})$ evaluated on $[M]$
SO_n	Special orthogonal group over the reals
$T(V)$	Tensor algebra of V, 3.1
T^*M	Cotangent bundle of a manifold M
T_pM	Tangent space of a manifold M at a point $p \in M$.
TM	Tangent bundle of a manifold M
trA	Trace of the matrix A

V	Vector space
$w_k(E)$	k-th Stiefel-Whitney class of a vector bundle E, C.5
$X \times_G F$	Fibre bundle associated to the principal $G-$bundle X with fiber F, A.5
$\mathrm{Pin}(V, q)$	Pin group of V with respect to q, 3.9
Pin_n	$\mathrm{Pin}(\mathbb{R}^n, q), q$ standard Euclidean form
$\mathrm{Spin}(V, q)$	Spin group of V with respect to q, 3.9
Spin_n	$\mathrm{Spin}(\mathbb{R}^n, q), q$ standard Euclidean form